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Unitary Equivalence and Saturation with Borel Complexity and Automorphisms of C^*-Algebras

التكافؤ الآحادي والتشبع مع تعقدية بورل والاوتومورفيزمات لجبريات- C^*

A Thesis Submitted in Fulfillment of the Requirement for the Degree of M.Sc. in Mathematics

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Dedication

To my Family.

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Abstract

The central sequences and inner derivations with regularity properties in the classification program for separable amenable $\boldsymbol{C}^{*}$-algebras are shown. We obtain the automorphisms of $C^{*}-$algebras and all Calkin algebras, similarly the unitary equivalence, and Rohlin property of automorphisms of separable C^{*}-algebras and Jiang–Su algebra. We determine the countable saturation and chain condition of Corona algebras and certain C^{*}-algebras and Banach algebras. We classify the strongly self- absorbing and descriptive set theory of C^{*}-algebra with model theory of operator algebras and Borel complexity.

الخلاصة

تم أيضاح المتتاليات المركزية والمشتقات الداخلية مع خصائص الانتظام في برنامج التصنيف لأجل جبريات$-C^*$ القابلة للانفصال. تم التحصيل على الاوتومورفيزم لجبريات$-C^*$ وكل جبريات كالكن وبالمثل التكافؤ الآحادي وخاصية روهلن للاوتومورفيزمات لجبريات$-C^*$ المنفصلة وجبر جيانق–سو. تم تحديد التشبع القابل للعد وشرط السلسلة لجبريات كورونا وجبريات$-C^*$ الأكيدة وجبريات باناخ. تم تصنيف الامتصاص الذاتي القوي ونظرية الفئة الموصوفة لجبر$-C^*$ مع نظرية النموذج وجبريات المؤثر وتعقدية بورل.

Introduction

We show that a separable C^*-algebra Ahas continuous trace if and only if each central sequence in A is trivial. This is used to show that the condition, that every derivation of A is determined by a multiplier of A, is equivalent to the condition that every summable central sequence in A is trivial. We shall extend the theorem of Lance and Smith to C^*-algebras of bundles whose fibres are C^*-algebras.

The Proper Forcing Axiom implies all automorphisms of every Calkin algebra associated with an infinite-dimensional complex Hilbert space and the ideal of compact operators are inner. Although our results were obtained by considering C^*algebras as models of the logic for metric structures, the reader is not required to have any knowledge of model theory of metric structures (or model theory, or logic in general). The proofs involve analysis of the extent of model-theoretic saturation of corona algebras. We study commutants modulo some normed ideal of n-tuples of operators which satisfy a certain approximate unit condition relative to the ideal.

We include a brief history of the program's successes since 1989, a more detailed look at the Villadsen-type algebras which have so dramatically changed the landscape, and a collection of announcements on the structure and properties of the Cuntz semigroup. This characterizes the Jiang–Su algebra $\mathcal{Z}$ as the uniquely determined initial object in the category of strongly self-absorbing C^*-algebras.

We establish the Borel computability of various C^*-algebra invariants, including the Elliott invariant and the Cuntz semigroup. This implies a dichotomy for the Borel complexity of the relation of unitary equivalence of automorphisms of a separable unital C^*-algebra: Such relation is either smooth or not even classifiable by countable structures.

We introduce a version of logic for metric structures suitable for applications to C^*-algebras and tracial von Neumann algebras. We introduce the countable chain condition for C^*- algebras and study its fundamental properties. We study the saturation properties of several classes of C^*-algebras. Saturation has been shown by Farah and Hart to unify the proofs of several properties of coronas of σ-unital C^*-algebras; we extend their results by showing that some coronas of non-σ-unital C^*-algebras are countably degree-1saturated. We then relate saturation of the abelian C^*-algebra $C(X)$, where X is 0-dimensional, to topological properties of X, particularly the saturation of $CL(X)$.

For projectionless C^*-algebras absorbing the Jiang–Su algebra tensorially, we study a kind of the Rohlin property for automorphisms. We show that the crossed products obtained by automorphisms with this Rohlin property also absorb the Jiang–Su algebra tensorially under amild technical condition on the C^*-algebras. We show that if A is $\mathcal{Z}$, $\mathcal{O}_2$, $\mathcal{O}_\infty$, a UHF algebra of infinite type, or the tensor product of a UHF algebra of infinite type and $\mathcal{O}_\infty$, then the conjugation action $\mathrm{Aut}(A) \curvearrowright \mathrm{Aut}(A)$ is generically turbulent for the point-norm topology.

The Contents

Chapter 1
Central Sequences and Automorphisms of C^*-Algebras

We show that an equivalent to a representation $A = A_1 \oplus A_2$, where A_1 has continuous trace and A_2 is the restricted direct sum of simple C^*-algebras.

Section (1.1): Inner Derivations of Separable C^*-Algebras

In 1968 S. Sakai [19] showed that every derivation of a simple C^*- algebra with unit is inner. Since then a fair amount of work, notably by G. A. Elliott [11]-[13], has been invested to find C^*- algebra with only inner derivations. It was soon apparent that in the case of a C^*- algebra A without unit the correct problem is: are there any derivations of A not of the form $\mathrm{ad}(h)$, where h is a multiplier of A ? (If A is separable and has only inner derivations, then by [3] $A = B \oplus C$, where B has a unit and C is commutative.) Since there is a bijective correspondence between derivations of A and derivations of its multiplier algebra $M(A)$, the problem can be for mulated as finding those C^*- algebras A for which $M(A)$ has only inner derivations. S. Sakai showed in [20] that such was the case for all simple C^*- algebras, and in [3] it was established for separable C^*- algebras with continuous trace.

With W. B. Arveson's theory of spectral subspaces a new set of ideas was introduced in operator algebra theory. It was used in [15] to show that each $*$_derivation of a algebra A has the form $\mathrm{ad}(ih)$, where $h \in A''$ (the enveloping von Neumann algebra of A; see [9]), and is the strong limit of an increasing net of positive operators in A. For example, in conjunction with Lemma(1.1.1) it gives the lifting theorem for derivations [18], and, as we shall see, it also provides the missing tool for solving the separable case of the above mentioned problem (see [3]).

The next lemma is straightforward and cannot be attributed to anybody. We prove it here because we shall use it repeatedly. It is used in the proof of the lifting theorem for derivations. Furthermore, taking the element x in the lemma to be an open central projection in A'' supporting a closed ideal I in $A(i.e. I\, xA'' \cap A)$, it shows the existence of approximate units for I which are quasi-central for A. Such approximate units have turned out to be rather useful; see [5] and [7]. Finally, it has been used recently by $G. A.$ Elliott to give a partial solution of our problem of classifying those separable C^*- algebras A for which $M(A)$ has only inner derivations (see [13]).

Lemma (1.1.1)[1]: Let A be a C^*- algebra and A''its enveloping von Neumann algebra. If $x \in A''$ and x derives A (i.e.,$xa - ax \in A$ for all a in A), then there is a net $\{x_\lambda\}$ in A converging strongly to x such that $\lim\|(x - x_\lambda)a - a(x - x_\lambda)\| = 0$ for each a in A. Moreover, $\{x_\lambda\}$can be chosen in the convex hull of any bounded net in A converging strongly to x.

Proof: Let $\{y_i \ |i \in 1\}$ be a bounded net in A converging strongly to x. (By Kaplansky's density theorem such a net exists.) Denote by Λ the net (with the obvious ordering) of triples$\lambda = \{i, \mu, \varepsilon\}$, where $i \in I, \mu,$ is a finite subset of Λ, say $\mu = \{a_1, a_2, \ldots, a_n\}$, and $\varepsilon > 0$. It suffices to show that for each λ there is a convex combination $x_\lambda = \sum \varepsilon_j y_j$, such that $i < j$ for all j and $\|(x - x_\lambda)a_k - a_k(x - x_\lambda)\| < \varepsilon$ for $1 \le k \le n$. Replacing A by its n-

fold direct sum and setting $a = (a_1, \ldots, a_n)$, we may assume $n = 1$ and thus reduce the problem to finding $x_\lambda = \sum \varepsilon_j y_j$, with $i < j$ such that $\|(x - x_\lambda)a_1 - a_1(x - x_\lambda)\| < \varepsilon$.

The set

$$E = \{y_j a_1 - a_1 y_j | i < j\} \subset A$$

Contains $xa_1 - a_1x$ as a σ-weake limit point, and $xa_1 - a_1x \in A$, since x derives A. Since A'' is isomorphic (as a Banach space) to the second dual A^{**} of A and the σ-weak topology on A is the $\sigma(A, A^*)$ - topology, it follows from the Hahn- Banach theorem that Conv E contains $xa_1 - a_1x$ as a limit point in norm, which we need.

We use Dixmier [9]. A will denote a separable C^* - algebra, $\hat{A}$ its spectrum and $\check{A}$ its primitive (or prime) spectrum , both equipped with the Jacobson topology. We denote by A'' the enveloping von Neumann algebra of A, and by $M(A)$ the C^* - algebra of multipliers of A in A''. Note that if $x \in A''$ and $xa_0 \in A, a_0x \in A$ for some strictly positive elements ao of A (see [2]), then $x\ E\ M(A)$. Recall from [6] that the strict topology (R. C. Busby's invention) on $M(A)$ is determined by the semi-norms $x \to \|x_a\| + \|a_x\|, a \in A$, and that for a bounded sequence (x_n) in $M(A)$ to converge strictly to zero it suffices that $\|x_n a_0\| + \|a_0 x_n\| \to 0$ for some strictly positive element a_0 in A. Finally we shall always denote the center of $M(A)$ by $Z(A)$, and we note that by the Dauns-Hofmann theorem (see [10] or [14]) we may identify $Z(A)$ with the algebra of bounded continuous functions on $\check{A}$ (or on $\hat{A}$).

A will denote a separable C^* - algebra. A bounded sequence (x_n) in A is central if Lim $\|ax_n - x_n a\| = 0$ for each a in A. Clearly (x_n^*) is a central sequence if (x_n) is, so that each central sequence in A is a combination of central sequences in A_{sa} . From the Stone-Weierstrass theorem it follows that if (x_n) is a central sequence in A_{sa}, then $(f(x_n))$ is a central sequence for each bounded continuous function $f: \mathbb{R} \to C$. In particular, each central sequence in A_{sa} is the difference of central sequences in A_+. We can therefore concentrate our attention on central sequences in A_+.

A central sequence (x_n) in $A_+$is summable if there is an element x in A'' such that $\sum x_n = x$ (strongly convergent sum).

A central sequence (x_n) in A is trivial if there is a sequence (z_n) in $Z(A)$ such that $(x_n - z_n)$ converges strictly to zero. Note that a central sequence (x_n) in $M(A)$converges strictly to zero provided that$\|x_n a\| \to 0$ for each a in A.

Lemma (1.1.2)[1]: Acentral sequence (x_n) in A is trivial if and only if there is a sequence (z_n) in $Z(A)$ with $\|z_n\| < \|x_n\|$ for all n such that $(x_n - z_n)$ tends strictly to zero.

Proof: Assume that (x_n) is trivial, and choose (y_n) in $Z(A)$ such that $(x_n - y_n) \to 0$strictly. Define fn in $C^b(\ C)$ by $f_n(\xi) = \xi\ if\ |\xi| \leq \|x_n\|, f_n(\xi) = \|x_n\| \xi |\xi|^{-1}$ otherwise. Put $z_n = f_n\ (y_n)$, and note that $z_n \in Z(A)$ with $\|z_n\| \leq \|x_n\|$.

If $(x_n - z_n)$ does not converge to zero strictly, there is an element a in A with $\|a\| = 1$ and $\varepsilon > 0$ such that, passing if necessary to a subsequence, we have $\|(x_n - z_n)a\| > \varepsilon$ for all n. Since $(x_n - z_n) \to 0$ strictly, we may further assume that $\|(x_n - z_n)a\| < \varepsilon / 2$for all n.

Choose for each n an irreducible representation π_n of A such that $\|\pi_n((x_n - z_n)a)\| > \varepsilon$. This implies that $\|\pi_n(a)\| > (\varepsilon/2)\|x_n\|^{-1}$. Furthermore

$$\|\pi_n(y_n - z_n)\| \geq \|\pi_n((y_n - z_n)a)\| \geq \|\pi_n((x_n - z_n)\, a)\| - \|\pi_n((x_n - y_n)a)\| > \varepsilon/2.$$

From the definition of z_n it follows that $\pi_n\,(y_n) = \lambda_n 1$, where $|\lambda_n| > \|x_n\| + \varepsilon/2$. Consequently

$\|\pi_n((x_n - y_n)a)\| \geq \|\pi_n\,(y_n a)\| - \|\pi_n\,(x_n a)\| \geq (|\lambda_n| - \|\pi_n\,(x_n)\|)\|\pi_n\,(a)\|(\varepsilon / 2)^2\|x_n\|^{-1}$.

Since $\|x_n\| \leq \alpha$ for some α and all n, we have shown that$\|(x_n - y_n)a\| > (\varepsilon / 2)^2\alpha^{-1}$ for all n, contradicting the assumption that $x_n - y_n \to 0$ strictly.W e must therefore have $x_n - z_n \to 0$ strictly, as desired.

Lemma (1.1.3)[1]: Assume that $\check{A}$ is a Hausdorff space. If every summable central sequence in A_+ is trivial, then every derivation of $M(A)$ is inner.

Proof: Let δ be a *-derivation of A and let δ also denote the unique extension to a derivation of $M(A)$By [15] there is a lower semi-continuous element h_0 in A_+''such that $\delta = i\ \mathrm{ad} h_0$. Take any $\varepsilon_0 > 0$, and note from [4] that if $h = h_0 + \varepsilon_0 1$, then $h \in (A_+)^m$, since $h_0 \in (((A_{sa})^m)^-)_+$ -Further, $\delta = i\ \mathrm{ad}\ h$. There is thereforea n increasing sequence (h_n) in A_+ such that $h_n \nearrow h$; and by Lemma (1.1.1) we may assume that $\|(h - h_n)a_k - a_k(h - h_n)\| < 2^{-n}$ for each n and every $k \leq n$, where (a_k) is a dense sequence in A.

Fix a in A. There are two cases:

(i) Some For $\varepsilon > 0$ and all n there are integers $i, j$with $i > j \geq n$such that$\|(h_i - h_i - z)a\| > \varepsilon$ for all z in $Z(A)$with $\|z\| < \|h_i - h_i\|$.
(ii) For every $\varepsilon > 0$ there exists an n such that for all integers i, j with $i > j \geq n$, there is some$z$ in $Z(A)$ such that $\|z\| \leq \|h_i - h_j\|$and $\|(h_i - h_i - Z)a\| \leq \varepsilon$.

In case (i) we can by induction find a subsequence

$$h_{j1} \leq h_{i1} \leq h_{i2} \leq h_{i2} \leq \dots$$

such that with $x_n = h_{i_n} - h_{j_n}$, we have $\|(x_n - z)a\| > \varepsilon$ for all z in $Z(A)$ with $\|z\| \leq \|x_n\|$. It follows from Lemma (1.1.2) that (x_n) is a non-trivials ummable centrals equence(with $\sum x_n \leq h$). The lemmaw ill be establishedw hen we have shown that if case (ii) occurs for all $a$in A, then $\delta$is inner in $M(A)$. Thus suppose case (ii) holds for all a in A.

Let $a_0$be a strictly positive element in A, and denote by f the function on $\check{A}$ given by $f(\pi) = \|\pi\,(a_0)\|$. Since $\check{A}$ is a locally compact Hausdorff space, it follows from [9] (cf. [9]) that f is continuous, whence $f \in C_0(\check{A})$ by [9]. Since a_0 is strictly positive in A, f is strictly positive in $C_0(\check{A})$.Applying spectral theory to f and identifying $C^b(\check{A})$ and $Z(A)$, we can therefore find a sequence (e_m) in $Z(A)_+$ with $\sum e_m = 1$, such that the support of each e_m is contained in the open set $\{\pi \in \check{A} | 1/(m+1) < f(\pi) < 1/(m+1).\}$ In particular, $\|\pi(a_0)\| > 1/(m+1)$ whenever $\pi(e_m) \neq 0$.

Fix m and take $a = e_m a_0$. Since case (ii) holds, we can then, by induction with $\varepsilon_n = 2^{-n}(m+1)^{-1}$, find a subsequence [again denoted by (h_n)] and a sequence (z_n) in $Z(A)$ such that

$$\|z_n\| < \|h_{n+1} - h_n\| \text{ and } \|(h_{n+1} - h_n - Z_n)e_m a_0\| \leq \varepsilon_n. \quad (1)$$

Put$y_n = \sum_{k=1}^{n-1} z_k$ $Zk$and $b_n = (h_n - y_n)e_m a_0$. Then (1) shows that I $\|b_{n+1} - b_n\|$,$\leq \varepsilon_n$, so that (bn) is norm convergent to an element in A. Since (h_n) is strongly convergent to (h), it follows that $(y_n e_m a_0)$ is strongly convergent to some element b and that $h e_m a_0 + b \in A$.

Without loss of generality we may assume that $\|\delta\| < 1$ and therefore that $\|h\| < 1$. Consequently $\|h_n\| < 1$ for all n, and by (1) also $\|z_n\| < 1$. Since

$$\|b_n - b_1\| \leq \sum_{k=1}^{n-1} 2^{-k}(m+1)^{-1} < (m+1)^{-1} ,$$

we obtain for each π in $\check{A}$

$\|\pi\,(y_n e_m a_0)\| < (m+1)^{-1} + \left\| \pi\left(\left(h_n - h_j + z_1\right)e_m a_0\right)\right\| < (m+1)^{-1} +$ $\|\pi\,(a_0)\|$. If $\pi(e_m) \neq 0$, this implies that

$$\|\pi\,(y_n e_m)\|\|\pi\,(a_0)\| = \|\pi\,(y_n e_m a_0)\| < \|\pi\,(a_0)\|.$$

From this we conclude that the sequence $(\,y_n e_m)$in $Z(A)$ is bounded by 4 and therefore has a σ-weak limit point fm in $A' \cap A$". From the preceding it follows that $b = f_m a_0$, so that $(he_m + f_m)a_0 \in A$. Moreover,

$$\|(he_m + f_m)a_o\| = \text{Lim}\|b_n\| \leq (m+1)^{-1} + \|b_1\| \leq (m+1)^{-1} + 2\|e_m a_0\| < (m+1)^{-1} + 2(m-1)^{-1}.$$

Since $e_n e_m = 0$ when $|n - m| > 1$, the same is true for fnm. Therefore the two sequences $\left((he_{2m} + f_{2m})a_0\right)$ and $\left((he_{2m} - + f_{2m-1})a_0\right)$ consist of pairwise orthogonal elements in A. Since their elements tend to zero, their sums belong to A; i.e., with $f = \sum f_m$ we have

$$(h+f)a_0 = \sum(he_{2m} + f_{2m})a_0 + (he_{2m-1} + f_{2m-1})a_0 \in A.$$

Since a_0 is strictly positive, it follows that $h + f \in M(A)$. As $f \in A' \cap A''$ we have $\delta = ad\big(i(h+f)\big)$, so that δ is inner in $M(A)$, completing the proof.

Lemma(1.1.4)[1]: (cf. [17]). Let $\hat{A}$ denote the spectrum of A equipped with the Jacobson topology, and use the open continuous surjection $\hat{A} \to \check{A}$ to embed $Z(A)$ in the von

Neumann algebra $\mathcal{F}(\hat{A})$ of all bounded functions on $\hat{A}$. Then $Z(A)^{-w} = \mathcal{F}(\hat{A})(^{-w}$ denoting weak closure) if and only if A is of type 1 and $\hat{A}$ is a Hausdorff space.

Proof: If A is of type I, then $\hat{A} = \check{A}$. If furthermore A is Hausdorff, we know from the Dauns-Hofmann theorem (see [10] or [14]) that $Z(A)$ separates the points in $\hat{A}$, whence $Z(A)^{-w} = \mathcal{F}(\hat{A})$.

Conversely, if $Z(A)^{-w} = \mathcal{F}(A)$, then $Z(A)$ separates the points in Â. It follows that $\check{A} = \hat{A}$, so that A is of type I by [9]. Furthermore $\hat{A}$ is a Hausdorff space, since $Z(A) = C^b(\hat{A})$.

Theorem (1.1.5)[1]: A separable C^*-algebra A has continuous trace if and only if each central sequence in A is trivial.

Proof: Suppose first that A has continuous trace, and let (x_n) be a central sequence in A. Let B denote the C^*-algebra consisting of all convergent sequences from A, i.e.

$$B = C(\mathbb{N}U\{\infty\})\otimes A.$$

Define a derivation $\delta$of B by $\delta = ad(x)$, where $x = (x_n)$. Note that if $b = (b_n) \in B$, then $b_n x_n - x_n b_n \to 0$so that $\delta(b) \in B$. Since B has continuous trace, there is by [3] an element h in $M(B)$ such that $\delta = ad(h)$. By [6] h has the form $(h_n)(1 \le n \le \infty)$, where $\{h_n\} \subset M(A)$ and $h_n \to h_\infty$ strictly. Let $Z_n = x_n - h_n$. Then $z_n \in Z(A)$, since $ad(x-h) = 0$. Moreover, if x_∞. denotes any σ-weak limit point of (x_n), then $x_\infty \in A' \cap A''$, since (x_n),is central. Further, since $z_n \in C\, Z(A)\ and\ (x_\infty - h_\infty)$ is a σ-weak limit point of (z_n) (because the strict topology is stronger than the σ-weak topology), then $(x_\infty - h_\infty) \in A' \cap A''$, so $h_\infty \in Z(A)$. Thus the sequence with elements

$$x_n - Z_n - h_\infty = h_{n-}h_\infty$$

Converges strictly to zero, proving that$(x_n)$is a trivial central sepuence.

For the converse assume first that $\hat{A}$ is a Hausdorff space (so that A is of type I). If A does not have continuous trace, there is an outer derivation of $M(A)$ by [3]. But then by Lemma(1.1.3) there is also a non-trivial central sequence in A_+(even a summable one).

If $\hat{A}$ is not a Hausdorff space, there is by Lemma(1.1.4) a characteristic function p on $\hat{A}$ which cannot be weakly approximated by elements in $Z(A)$. Since $\mathcal{F}(\hat{A})$ is the center of the weak closure of A in its atomic representation, we may regard p as a central projection in A''. Let U be a σ-weak neighbor-hood of 0 in A'' with the σ-weak closure of $U - U$ disjoint from $Z(A) + p$. Choose a net (x_λ) in the unit ball of $A \cap (U + p)$ which is σ-wealdy convergent to p. By Lemma (1.1.1) and the separability of A we may choose a sequence (x_n) in the convex hull of (x_λ)such that $\|x_n a - a x_n\| \to 0$ for each a in A, i.e., $(x_n)$is a central sequence. If it was trivial, then $x_n, - z_n, \to 0$ strictly for some sequence (z_n) in the unit ball of $Z(A)$ by Lemma (1.1.2) Since the strict topology is stronger than the σ-weak topology and $(x_n) \subset U + p$, it follows that $z_n + p \in U - U$ eventually. This contradicts our choice of p and shows that (x_n) is non-trivial.

The following two conditions on a separable C^*-algebra A will occur repeatedly.

(A) Every summable central sequence in A_+ is trivial.

(B) Every derivation of $M(A)$ is inner.

Lemma (1.1.6)[1]: If A satisfies condition (A) and $Z(A) = \mathbb{C}$, then A is simple.

Proof: Assume, to obtain a contradiction, that I is a non-trivial closed ideal of A, and let p denote the open central projection in A'' supporting I, i.e. $I = pA'' \cap A$. If b_0 is a strictly positive element in I, then p is the range projection of b_0 in A''. By spectral theory we can therefore find an increasing sequence (h_n) in the C^*-algebra generated by b_0 such that $0 \le h_n \le 1, h_{n+1}h_n = h_n$ for all n, and $h_n \nearrow p$. Using Lemma (1.1.1) we can further assume that (h_n) is a central sequence. In particular we may assume that $\|h_n a_0 - a_0 h_n\| < 2^{-n}$ for all n, where a_0 is a strictly positive element in A. Put

$$x_n = h_{2n} - h_{2n-I} \text{ and } y_n = h_{2n-1} - h_{2n-2} \text{ (with } h_0 = 0).$$

Then (x_n) is a summable central sequence in A_+ with $\|x_n\| = 1$ for all n. Furthermore $x_n x_m = 0$ if $n \neq m$. The same statements hold for the sequence (y_n).

By assumption (x_n) is trivial so that for some sequence λ_n in $\mathbb{C}$ [$= Z(A)$] we have $x_n - \lambda_n \longrightarrow O$ strictly. However, this implies that

$$|\lambda_n| = \|\lambda_n x_1\| = \|(x_1 - \lambda_n)x_1\| \longrightarrow 0,$$

so that $x_n \to 0$ strictly. In particular, $x_n a_0 \to 0$. Consider the partial sum $S_{nm} = \sum_{k=n}^{m} x_k a_0^{1/2}$, where $n < m$. Then

$$\begin{aligned}\|S_{nm}\|^2 &= \left\| \sum_{n \le k,l < m} x_k a_0 x_1 \right\| \\ &\le \left\| \sum_{n \le k \le m} x_k a_0 x_k \right\| + \left\| \sum_{n \le l < m} \left(\left(\sum_{k \neq l} x_k \right) (a_0 x_1 - x_1 a_0) \right) \right\| \\ &\le \sup_{n \le k} \|x_k a_0 x_k\| + \sum_{n \le l} \|a_0 x_1 - x_1 a_0\|\end{aligned}$$

which tends to zero as $n \to \infty$. Since A is complete, it follows that $\sum_{x_n} a_0^{1/2} \in A$. The exact same reasoning on (y_n) shows that ,$\sum_{y_n} a_0^{1/2} \in A$. But then

$$pa_0 = \left(\sum x_n a_0^{1/2} + y_n a_0^{1/2} \right) a_0^{1/2} \in A,$$

and, since a_0 is strictly positive, this implies that $p \in M(A)$. But p is central, so $p \in Z(A) (= \mathbb{C})$. This contradicts the non-triviality of I.

Lemma(1.1.7)[1]: If A is primitive and satisfies condition (B), then A is simple.

Proof: If A is non-simple, we construct the orthogonal central sequences (x_n) and (y_n) as in the proof of Lemma (1.1.6) If a_0 is a strictly positive element in A and both $x_n a_0 \to$

0and $y_n a_0 \to 0$, we conclude as in the proof of Lemma(1.1.6)that $p \in Z(A)$. However, $Z(A) = \mathbb{C}1$, since A is primitive, a contradiction. Thus we may assume, passing if necessary to a subsequence of $(x_n)\big(or\ (y_n)\big)$ that for some $\varepsilon > 0$ we have $\|x_n a_o\| > \varepsilon$ for all n. Furthermore, we may assume that $\|x_n a_o - a_o x_n\| < 2^{-n}$ for each n and all $k \le n$, where (a_k) is a dense sequence in A.

Let $\Lambda = (\lambda_n)$ be a sequence of zeros and ones. The element $\sum \lambda_n x_n$derives A, and thus by (B) there is some z in $A'' \cap A'$ such that $z + \sum \lambda_n x_n \in M(A)$. Since A is primitive, it has a faithful irreducible representation $\pi, i.e. \pi(A'' \cap A') = \mathbb{C}1 = \pi\big(Z(A)\big)$. It follows that $\pi z + (\sum \lambda_n x_n) \in \pi\big(M(A)\big)$.

Put $a_\Lambda = \pi(\sum \lambda_n x_n a_0) \in \pi(A)$. If $\Lambda \neq \Lambda'$, say $\lambda_n \neq \lambda'_n$,then

$$\|a_\Lambda - a_{\Lambda'}\| \ge \left\|x_n \sum (\lambda_k - \lambda'_k) x_n a_0\right\| = \|x_n^2 a_0\| \ge \|a_0 x_n^2 a_0\| > \varepsilon^2.$$

Since there are uncountablymany $\Lambda's$,this contradicts the separability of A.

Lemma (1.1.8)[1]: If A is simple, then it satisfies condition (A).

Proof: Assume to obtain a contradiction that (x_n) is a non-trivial summa-ble central sequence in A_+. Passing to a subsequence, we may assume that for some $\varepsilon > 0$ and all n we have $\left\|x_n^{1/2} a_0\right\| > \varepsilon$, where a_0 is strictly positive in A, and that $\|x_n a_k - a_k x_n\| < 2^{-n}$for each n and all $k \le n$, where (a_k) is a dense sequence in A. If $\Lambda = (\lambda_n)$is any sequence of zeros and ones, then the element $\sum \lambda_n x_n$ belongs to A" and derives A. Since A is simple, it follows from Sakai's theorem (see [19], [20] or [3]) that $\sum \lambda_n x_n \in M(A)$. Thus $a_\Lambda = \sum \lambda_n a_0 x_n a_0 \in A$.

Put $b_n = \sum_{k>n} a_0 x_n a_0$. Then $(b_n) \subset A_+$ and $b_n \to O$ strongly. We can there-fore assume, passing if necessary to a subsequence of (x_n), that for each n there is a state φ_n of A with .

$$\varphi_n(a_0 x_n a_0) = \|a_0 x_n a_0\| \left(= \left\|x_n^{1/2} a_0\right\|^2 \ge \varepsilon^2\right),$$

such that $\varphi_n(b_n) < \frac{1}{2}\varepsilon^2$. If $\Lambda \neq A'$, let n be the first number with $\lambda_n \neq \lambda'_n$. Then

$$\|a_\Lambda - a_{\Lambda'}\| \ge |\varphi_\Lambda(a_\Lambda - a_{\Lambda'})|$$

$$\ge \varphi_n(a_0 x_n a_0) - \varphi_n(b_n) > \frac{1}{2}\varepsilon^2$$

This contradicts the separability of A.

Lemma(1.1.9)[1]: Let $\pi: A \to B$ be a surjectice morphism between separable C^*-algebras A and B. If (y_n) is a central sequence in B, there is a central sequence (x_n) in A with $\pi(x_n) = y_n$. If (y_n) is summable, (x_n) can be chosen to be summable. If (y_n) is non-trivial, (x_n) is automatically non-trivial.

Proof: (cf. the proof of [18]). Given a central sequence (y_n) in B, choose by [6] a sequence (b_n) in A with $\pi(b_n) = y_n$ and $\|b_n\| = \|y_n\|$f or all n. Let (a_k) be a dense sequence in A, and choose by [5] or [7] a quasi-central approximate unit $\{u_\lambda\}$ for $\ker\pi$. For each a in A we have, by (1) in [9],

$$\lim_{\lambda} \|(1-u_\lambda)b_n a - a(1-u_\lambda)b_n\| = \lim_{\lambda} \|(1-u_\lambda)(b_n a - a b_n)\|$$

$$= \|y_n\pi(a) - \pi(a)y_n\|.$$

We can therefore choose λ such that with $x_n = (1-u_\lambda)b_n$ we have

$$\|x_n a_k - a_k x_n\| < \|y_n\pi(a_k) - \pi(a_k)y_n\| + 2^{-n}$$

for all $k \leq n$. It follows that (x_n) is a central sequence for A.

If $(y_n) \subset B_+$ is summable, say $\sum y_n \leq 1$, we use [16] and induction to find $(b_n) \subset A_+$, with $\sum_{k=1}^{n} b_k \leq 1$for every n. Then we define $x_n = b_n^{1/2}(1-u_\lambda)b_n^{1/2}$ for a suitable λ and obtain as before a central sequence (x_n), which is now also summable, since $x_n \leq b_n$ for all n.

If (y_n) is non-trivial, then $(x_n)$is non-trivial, since $\pi(Z(A)) \subset Z(B)$.

Proposition (1.1.10)[1]: If a separable C^*-algebra A satisfies condition (A) or (B), then $\check{A}$ is a T_1-space (i.e., points are closed).

Proof: Let π be an irreducible representation of A. By Lemma (1.1.9), $\pi(A)$ satisfies condition (A) if A does, and, by the lifting theorem for derivations [18], $\pi(A)$ satisfies condition (B) if A does. Thus either Lemma (1.1.6) or Lemma (1.1.7) applies to show that $\pi(A)$is simple. Consequently every primitive ideal of A is maximal, i.e. $\check{A}$ is a T_1-space. Recall from [8] that a point g in a T_1-space X is separated if for each π'in $X, \pi' \neq \pi$, there are disjoint neighborhoods of π and π'. Thus X is a Hausdorff (= separated) space precisely when every point is separated. If A is a separable C^*-algebra (and $\check{A}$ is a T_1-space), then the separated points in $\check{A}$ form a dense set by [8].

Lemma(1.1.11)[1]: Assume that $\check{A}$ is a T_1-space. Let (π_n) be a convergent sequence of distinct, separated points in $\check{A}$, and denote by F the closed set of limit points for (π_n). Let B be the quotient of A corresponding to F $\left(i.e., \check{B} = F\right)$. If B_+ contains a non-trivial central sequence, then A satisfies neither condition (A)nor (B).

Proof: If B_+ contains a non-trivial central sequence, there is by Lemma (1.1.9) a non-trivial central sequence (x_n) in B_+. Let a_0 be a strictly positive element in A, and denote by ρ the quotient map $\rho: A \to B$. Passing to a subsequence, we find an $\varepsilon > 0$ such that for every z in $Z(A)$ and all n we have $\|\rho((x_n - z)a_0)\| > \varepsilon$.

We claim that for each n and each k_0 there is a $k \geq k_0$ such that $\|\pi_k((x_n-\lambda)a_0)\| > \varepsilon/2$ for all λ in $\mathbb{C}$.Otherwise we have $(\lambda_k) \subset \mathbb{C}$ such that for all $k \geq k_0$, $\|\pi_k((x_n - \lambda_k)a_0)\| \leq \varepsilon/2$. If $\pi \in F$, then $\pi(a_0) \neq 0$, and so eventually $\|\pi_k(a_0)\| > \frac{1}{2}\|\pi(a_0)\|$. It follows that (λ_k) is bounded, and, passing to a sub-sequence, we may assume that $\lambda_k - \lambda \in \mathbb{C}$. But then the closed set

$\{\pi \in \check{A} | \|\pi((x_n - \lambda)\, a_0)\| \leq \frac{2}{3}\varepsilon\}$contains$(\pi_k)$ and therefore also F, and

$$\|\rho((x_n - \lambda)\, a_0)\| = \underset{\pi \in F}{Sup} \|\pi((x_n - \lambda)\, a_0)\| \leq \frac{2}{3}\varepsilon,$$

a contradiction. Passing to a subsequence of (π_k), we may therefore assume that $\|\pi_n((x_n - \lambda)a_0)\| > \frac{1}{2}\varepsilon$ for all λ in $\mathbb{C}$ and all n.

Take πin F. Since π_1 is a separated point, there are disjoint open neighborhoods G_1 and $G_1'$of π_1, and π. Since $\pi_n \to \pi$, we have eventually $\pi_n, \in G_1'$. Continuing by induction and passing to a subsequence of (π_n), we find a sequence (G_n) of pairwise disjoint open sets in $\check{A}$ such that (after relabeling) $\pi_n \in G_n$.

Let I_n be the non-zero closed ideal of A corresponding to $G_n(i.e. \check{I}_n = G_n)$. Since $G_n \cap G_m = \emptyset$, we have $I_n \cap I_m = \{O\}$ for $n \neq m$. Choose a quasi-central approximate unit $\{u_\lambda\}$ for I_n, and let $\{a_k\}$ be a dense sequence in A. Assuming, as we may, that $\|x_n a_k - a_k x_n\| < 2^{-n}$ for $k \leq n$, we put $y_n = u_\lambda x_\lambda n_\lambda$ for λ so large that $\|y_n a_k - a_k y_n\| < 2^{-n}$for $k \leq n$ and $\|\pi_n((x_n - y_n)\, a_0)\| < \frac{1}{4}\varepsilon$.

The central sequence (y_n) in A_+ is summable, since $y_n y_m = 0$ for $n \neq m$, and non-trivial, since for each z in $Z(A)$.

$$\|(y_n - z)a_0\| \geq \left\|\pi_n\left(((y_n - z))a_0\right)\right\| \geq \|\pi_n((x_n - \lambda)a_0)\| - \|\pi_n((y_n - x_n)a_0)\| > \frac{1}{4}\varepsilon,$$

where $\lambda 1 = \pi_n(z)$. Thus condition (A) is violated. Suppose that for each sequence $\Lambda = (\lambda_n)$ of zeros and ones, the deriver $\sum \lambda_n y_n$ of A [and hence of $M(A)$] gives an inner derivation of $M(A)$. Then $z_\Lambda + \sum \lambda_n y_n \in M(A)$ for some $z_\Lambda$in $A'' \cap A'$, and we define $a_\Lambda = (z_\Lambda + \sum \lambda_n y_n)a_0$ in A. If $\Lambda \neq \Lambda'$, say $\lambda_n > \lambda_n'$, we have

$$\|a_\Lambda - a_{\Lambda'}\| \geq \|\pi_n((a_\Lambda - a_{\Lambda'} + (\lambda_n - \lambda_n')y_n)a_0)\|$$
$$\geq \|\pi_n((\lambda + y_n)\, a_0)\| > \frac{1}{4}\varepsilon,$$

where $\lambda 1 = \pi_n(a_\Lambda - a_{\Lambda'})$. This contradicts the separability of A, and proves that $M(A)$ has outer derivations, in violation of condition (B).

Lemma(1.1.12): As in Lemma (1.1.11), let (π_n) be given and define $F$and B. If $Z(B) \neq \mathbb{C}$, then A satisfies neither condition (A) nor (B).

Proof: (cf. the proof of [3]). Let p denote the quotient map $\rho: A \to B$. Since A is separable, we have $\rho(M(A)) = M(B)$ by [6]; thus by assumption there exists x in $M(A), 0 \leq x \leq 1$, such that $\rho(x) \in Z(B)$ with $\rho_0(x) = 0, \rho_1(x) = I$for ρ_0, ρ_1, in F. Let a_0 be a strictly positive element in A, and assume that $\|a_0\| \leq 1$ and $\|\rho_0(a_0)\| = \|\rho_1(a_0)\| =$

1.We claim that eventually$\left\|\pi_n\left((x-\lambda)a_0\right)\right\| \geq \frac{I}{3}$for all λ in $\mathbb{C}$. Otherwise there would exist a bounded sequence (λ_n) such that $\left\|\pi_n\left((x-\lambda_n)a_0\right)\right\| \geq \frac{I}{3}$for all $n \geq n_0$. Passing to a subsequence, we may assume that $\lambda_n \rightarrow \lambda$ and that $\left\|\pi_n\left((x-\lambda)a_0\right)\right\| \leq \frac{1}{3}$for all n. But then the same is true for the limit points; in particular $\left\|\rho_0\left((x-\lambda)a_0\right)\right\| \leq \frac{1}{3}$, $\left\|\rho_0\left((x-\lambda)a_0\right)\right\| \leq \frac{1}{3}$.

With our choice of x this implies that $|\lambda| \leq \frac{1}{3}$and $|1-\lambda| \leq \frac{1}{3}$, a contradiction.

Let (a_k) be a dense sequence in A.For each m the set

$$K_m = \underset{k \leq m}{U} \left\{\pi \in \check{A} \| \pi(xa_k - a_k x)\| \geq 2^{-m}\right\}$$

is compact by [9] and disjoint from F. Since (π_n) consists of separated points, each set

$$F_n = \{\pi_n | k \geq n\} \, U \, F$$

is closed in $\check{A}$; and $\cap_n K_m \cap F_n = \emptyset$. Consequently $K_m \cap F_n = \emptyset$. for some n, and passing to a subsequence of (π_n), we may assume that $\|\pi_n\,(xa_k - a_k x)\| < 2^{-n}$for all $k \leq n$. As in the proof of Lemma(1.1.11), choose a sequence (G_n) of pairwise disjoint open subsets of $\check{A}$ with $\pi_n \in G_n$, and let I_n denote the closed ideal of A corresponding to G_n.

Fix n, and let $\{u_\lambda\}$ be a quasi-central approximate unit for ker π_n. Then $\lim_\lambda \left\|x^{\frac{1}{2}}(1-u_\lambda)x^{\frac{1}{2}}a_k - a_k x^{\frac{1}{2}}(1-u_\lambda)x^{\frac{1}{2}}\right\| = \lim_\lambda \left\|x^{\frac{1}{2}}(1-u_\lambda)(xa_k - a_k x)\right\| = \|\pi_n(\,xa_k - a_k x)\|$,

by [9]. For sufficiently large λ we define $x_n = x^{1/2}(1-u_\lambda)x^{1/2}$ and have $\|x_n a_k - a_k x_n\| < 2^{-n}$ for all $k \leq n$. Let $\{v_\lambda\}$ be a quasi-central approximate unit for I_n, and, for sufficiently large λ, define $y_n = v_\lambda x_n v_\lambda$ to obtain

$$\|y_n a_k - a_k y_n\| < 2^{-n}\,, k \leq n, \tag{2}$$

and $\left\|\pi_n\left((x_n - y_n)a_0\right)\right\| < \frac{1}{6}$. This last inequality implies that for each λ in $\mathbb{C}$ we have

$$\begin{aligned}\left\|\pi_n\left((y_n-\lambda)a_0\right)\right\| &\geq \left\|\pi_n\left((x_n-\lambda)a_0\right)\right\| - \left\|\pi_n\left((x_n-y_n))a_0\right)\right\| \\ &= \left\|\pi_n\left((x-\lambda)a_0\right)\right\| - \frac{1}{6} \geq \frac{1}{6}\end{aligned} \tag{3}$$

Given (2) and (3) we can now show, exactly as in the proof of Lemma (1.1.11), that A has a non-trivial summable central sequence [viz. (y_n)] and that $M(A)$ has an outer derivation [of the form ad$(\sum \lambda_n y_n)$]].

Proposition(1.1.13)[1]: If a separable C^*-algebra A satisfies condition (A) or (B), then $\check{A}$ is a Hausdorff space.

Proof: From Proposition (1.1.10) we know that $\check{A}$ is a T_1-space. Assume, to obtain a contradiction, that $\check{A}$ is not a Hausdorff space. There are then at least two points ρ_0, ρ_1 in $\check{A}$ that cannot be separated. Since ρ_0 is not an isolated point and since the separated points are dense in $\check{A}$ by [8], we can find a sequence (the Jacobson topology is second countable when A is separable) of distinct separated points π_n in A such that (π_n) converges to ρ_0. The set F of limit points of (π_n) contains at least two points (viz. $(\pi_n)$and ρ_1), so the quotient B of A corresponding to F is not simple.

If B satisfies condition (A), then $Z(B) \neq \mathbb{C}$ by Lemma (1.1.6), and thus A will satisfy neither condition (A) nor (B) by Lemma(1.1.12) If, on the other hand, B does not satisfy condition (A), then A will satisfy neither condition (A) nor (B) by Lemma (1.1.11) We have found the desired contradiction.

Theorem(1.1.14)[1]: The following three conditions on a separable C^*-algebra A are equivalent:

(i) Every summable central sequence in A_+ is trivial.

(ii) Every derivation of $M(A)$ is inner.

(iii) $A = A_1 \oplus A_2$, where A_1 has continuous trace and A_2 is discrete (i.e., A_2 is the restricted direct sum of simple C^*-algebras).

Proof: $(A) \Rightarrow (B)$: Combine Proposition (1.1.13) with Lemma (1.1.3)

$(B) \Rightarrow (A)$: If π is an irreducible representation of A corresponding to a non-isolated point in $\check{A}$, then, since $\check{A}$ is a Hausdorff space by Proposition (1.1.13), it follows from Lemma(1.1.11) that $\pi(A)$ has no non-trivial central sequences. Since $\pi(A)$ is primitive, it follows from Theorem (1.1.5) that $\pi(A)$is isomorphic to the compact operators on a separable Hilbert space. Let G_1denote the open set in $\check{A}$ corresponding to the largest CCR ideal in A (cf. [9]), and let Go denote the set of isolated points in $\check{A}$.

From the first part of the proof we see that $G_0 \cup G_1 = \check{A}. Set\ G_2 = \check{A} \backslash G_1$. Then G_2 is closed, but, since it consists of isolated points, it is also open. Thus $\check{A} = G_1 \cup G_2$ (disjoint union). Let A_1 and A_2 be the direct summands of A corresponding to G_1 and G_2, respectively. Then A_1 is a CCR algebra with Hausdorff spectrum and satisfies condition (B). It follows from [3] that A_1 has continuous trace. Since $\check{A}_2$ $(= G_2)$ is discrete, A_2 is the restricted direct sum (cf. [9]) of simple C^*-algebras.

$(C) \Rightarrow (A)$: $If\ A = A_1 \oplus A_2$ whereA_1 has continuous trace and $A_2 = \bigoplus_0 B_k$, where (B_k) is a sequence of simple C^*-algebras, then each summable central sequence (x_n) in A_+ breaks into a sequence $(x_n^k), 0 \leq k < \infty$, of summable central sequences, where $(x_n^0) \subset A_1$ and $(x_n^k) \subset B_1$. From Theorem (1.1.5) we know that (x_n^0) is trivial, and by Lemma (1.1.8) each (x_n^k) is also trivial. Since $M(A) = M(A_1) \oplus \bigoplus M(B_k)$ (full direct sum), it follows that (x_n) is trivial, as desired.

Corollary (1.1.15)[1]: Let A be a separable C^*-algebra with unit, and assume that A has only inner derivations. Then A is the direct sum of a finite number of C^*sub-algebras which are either homogeneous of finite degree or simple. As pointed out in [13], the implications $(A) \Rightarrow (B)$and $(B) \Rightarrow (C)$ do not generally hold when A is allowed to be non-separable.

Corollary(1.1.16)[370]: Let $\pi: A \to A+\epsilon$ be a surjectice morphism between separable C^*-algebras A and $A+\epsilon$. If (y_n^m) is a central sequence in $A+\epsilon$, there is a central sequence (x_n^m) in A with $\sum_m \pi(x_n^m) = \sum_m y_n^m$. If (y_n^m) is summable, (x_n^m) can be chosen to be summable. If (y_n^m) is non-trivial, (x_n^m) is automatically non-trivial.

Proof: (cf. the proof of [18]). Given a central sequence (y_n^m) in B, choose by [6] a sequence (b_n^m) in A with $\sum_m \pi(b_n^m) = \sum_m y_n^m$ and $\sum_m \|b_n^m\| = \sum_m \|y_n^m\|$f or all n. Let (a_k) be a dense sequence in A, and choose by [5] or [7] a quasi-central approximate unit $\{u_\lambda^m\}$ for kerπ. For each a in A we have, by (1) in [9],

$$\sum_m \lim_\lambda \|(1-u_\lambda^m)b_n^m a - a(1-u_\lambda^m)b_n^m\| = \sum_m \lim_\lambda \|(1-u_\lambda^m)(b_n^m a - ab_n^m)\|$$

$$= \sum_m \|y_n^m \pi(a) - \pi(a) y_n^m\|.$$

We can therefore choose λ such that with $x_n^m = (1-u_\lambda^m)b_n^m$ we have

$$\sum_m \|x_n^m a_k - a_k x_n^m\| < \sum_m \|y_n^m \pi(a_k) - \pi(a_k) y_n^m\| + 2^{-n}$$

for all $k \le n$. It follows that (x_n^m) is a central sequence for A.

If $(y_n^m) \subset A_+ + \epsilon$ is summable, say $\sum_m \sum y_n^m \le 1$, we use [16] and induction to find $(b_n^m) \subset A_+$, with $\sum_{k=1}^n \sum_m b_k^m \le 1$ for every n. Then we define $\sum_m x_n^m = \sum_m b_n^{m/2}(1-u_\lambda^m)b_n^{m/2}$ for a suitable λ and obtain as before a central sequence (x_n^m), which is now also summable, since $x_n^m \le b_n^m$ for all n.

Section (1.2): Second Cech Cohomology

Let A be a C^*-algebra with identity, and let Aut A be the group of all *-automorphisms of A endowed with the norm topology. In their systematic investigation [42] of Aut A, Kadison and Ringrose considered the subgroups Inn A, $\gamma(A)$, $\Pi(A)$ of Aut A consisting of, respectively, inner automorphisms, those path-connected to the identity, and n-inner automorphisms. As well as proving some general theorems about the relationships between these subgroups, they investigated in detail some particular cases, including the algebra $C(X, M_n(C))$ of continuous functions from a compact space into the matrix algebra $M_n(C)$.Subsequently Lance [45] and Smith [52] considered the algebra $A = C(X, B(H))$, where X is compact and separable, and B(H) is the algebra of all operators on a Hilbert space H of dimension $\varkappa_0$. They proved the striking result that $\pi(A)/\gamma(A) \cong H^2(X, Z)$, the second $\check{C}$ech cohomology group of X with integral coefficients. It follows from their results that in this case $\Pi(A)$ coincides with the group $\mathrm{Aut}_{C(X)}A$ of automorphisms which preserve the $C(X)$-module structure of A, and that $\gamma(A) = \mathrm{Inn}$ A; hence their theorem identifies the group $Aut_{C(X)}A$ /Inn A of outer $C(X)$-auto-morphisms of $A = C(X, B(H))$ with $H^2(X, Z)$.

We shall extend the theorem of Lance and Smith to C^*-algebras of bundles whose fibres are C^*-algebras. We consider two distinct types of bundles: the first have as fibre the algebra K(H) of compact operators on a Hilbert space H, and as structure group Aut K(H) equipped with the topology of pointwise convergence; the algebra $A = \Gamma_0(E)$ of such a bundle E over a locally compact space X is called a stable continuous trace C^*-algebra with spectrum X. We show that if Inn A now denotes the group of automorphisms which are implemented by multipliers, then there is a short exact sequence

$$0 \rightarrow Inn\, A \rightarrow \mathrm{Aut}_{C_b(X)} A \xrightarrow{\eta} H^2(X, Z) \rightarrow 0.$$

In fact, if A is any separable continuous trace C^*-algebra then we can construct the homomorphism η, but it is not necessarily surjective. The second type of bundle we consider has as fibre a C^*-algebra B with identity and structure group Inn B in its norm topology. If E is such a bundle over a separable compact space X, with fibre B a von Neumann algebra factor, and if $A = \Gamma(E)$, then we obtain an exact sequence

$$0 \rightarrow InnA \rightarrow \pi(A) \xrightarrow{\eta} H^2(X, Z);$$

if in addition the unitary group of B is contractible, then η is surjective. In case of the trivial bundle $E = X \times B(H)$ we recover the theorem of Lance and Smith. The construction of η in both cases is a modification for non-trivial bundles of Lance's proof of [45, Theorem 4.3].

If E is a bundle of matrix algebras over a compact space X, the algebra $A = \Gamma(E)$ is called (by algebraists) an Azumaya algebra over $C(X)$. The exact sequence

$$0 \rightarrow Inn\, A \rightarrow \mathrm{Aut}_{C(X)} A \rightarrow Pic\, C(X) \cong H^2(X, Z)$$

is due to Rosenberg and Zelinsky [50], and a theorem of Knus [44] says that the range of ξ is contained in the torsion subgroup of $H^2(X, Z)$. A standard construction (cf. [39]) associates to each Azumaya algebra A over $C(X)$ an element $\delta(A)$ of the torsion subgroup of $H^3(X, Z)$, and in fact a theorem of Serre [39] asserts that every element of $H^3(X, Z)$ of finite order arises this way. Now Dixmier and Douady [34] have proved that stable continuous trace C^*-algebras with spectrum X are classified up to isomorphism by $H^3(X, Z)$; our result on stable continuous trace C^* −algebras shows that $H^2(X, Z)$ classifies the outer $C(X)$-automorphisms of such C^*-algebras. Thus analogies of the Serre-Knus results are valid for stable continuous trace C^*-algebras with the torsion subgroups replaced by the whole of the cohomology groups.

We show some technical results. The second part contains our results on automorphisms and derivations of a separable continuous trace C^*-algebra A. As well as the main theorem which we described above, we investigate the group of all outer automorphisms of A and discuss the relationship of the work with that of Kadison and Ringrose [42] and Brown, Green and Rieffel [28]. We concerned with bundles of C^*-algebras where the structure group has the norm topology. In addition to the theorem mentioned above, we look at what happens when the fibre has non-trivial centre, and show that the Dixmier-Douady classification of stable continuous trace C^*-algebras works also for these bundles.

We shall denote by $B(H)$ the C^*-algebra of all bounded linear operators on a separable Hilbert space H, by $K(H)$ the C^*-algebra of all compact operators on H, and by $U(H)$ the group of unitary operators on H. Unless we specifically say otherwise, all homomorphisms between C^*-algebras will be*-homomorphisms; this applies in particular to automorphisms and representations. If A is a C^*-algebra with identity, we shall denote the identity of A by 1 and the identity mapping: A → A by id. The group of all automorphisms of a C^*-algebra A will be denoted by Aut A, the centre of A by Z(A), and the group of unitary elements of A by U(A). We shall write A for the spectrum of A equipped with the Jacobson topology (see [32, Chapter 3]).

Let A be a C^*-algebra and let M(A) be its multiplier algebra—the collection of all pairs $m = (m', m'')$ of maps from A to A satisfying

$$am'\{b) = m''(a)b \quad \text{for } a, b \in A;$$

intuitively, m' and m'' represent left and right multiplication by the element m, and we usually write ma for $m'(a)$ and am for $m''(a)$. The collection M(A) is a C^*-algebra with identity containing A as a closed two-sided ideal, and has the following universal property: if B is a C^*-algebra containing A as a closed two-sided ideal such that $bA = 0$ implies $b = 0$, then there is an embedding of B into M(A) (see [30, Sections 2 and 3] for details). If $\alpha \in Aut\,A$ has the form $\alpha(a) = uau^*$ for some $u \in U(M(A))$ then we call α an inner automorphism, and we write $\alpha = Adu$ these form a subgroup of Aut A which we denote by Inn A. If the algebra A has an identity, then $M(A) = A$ and this coincides with the usual notion of inner.

Following Kadison and Ringrose [42] we denote by π(A) the group of automorphisms of A which are weakly inner in every faithful representation, and we call these ir-inner automorphisms. If B is a commutative subalgebra of M(A), then $Aut_B A$ will denote the collection of automorphisms of A which commute with the multipliers in B.

Let E be a continuous field of C^*-algebras over a locally compact(Hausdorff) space; that is, E is a parametrised family $\{E_x : x \in X$ of C^*-algebras together with a family Γ(E) satisfying

(i) Γ(E) is a ∗-algebra;
(ii) $\{a(x) : a \in \Gamma(E)\} = E_x$ for each $x \in X$;
(iii) for each $a \in \Gamma(E), x \to \|a(x)\|$ is continuous;
(iv) Γ(E) is closed under local uniform convergence.

The space $A = \Gamma_0(E)$ of continuous which vanish at infinity is a C^*-algebra in the uniform norm, and is also a module over the ring $C_b(X)$ of bounded continuous functions on X. An isomorphism $\phi: E \to F$ of fields over X is a collection of isomorphism $\phi_x: E_x \to F_x$ which carries Γ(E) onto Γ(E); we denote the induced isomorphism of $\Gamma_0(E)$ onto $\Gamma_0(F)$ by ϕ_*, and note that the correspondence $\phi \to \phi_*$ is functorial. See [32, Chapter 10].

Lemma(1.2.1)[21]: Let E be a continuous field of C^*-algebras over a locally compact space X, let $A = \Gamma_0(E)$ and let $\alpha \in Aut_{C_{b(X)}} A$.

(i) If $a_1, a_2 \in A$ satisfy $a_1(x) = a_2(x)$ for some $x \in X$, then $\alpha(a_1)(x) = \alpha(a_2)(x)$.
(ii) If Y is a compact subset of X, then α induces an automorphism α_Y of $\Gamma(E|_Y)$ such that $\alpha_Y(a|_Y) = \alpha(a)|_Y$ for $a \in A$.

Proof: For suppose $a_1(x) = a_2(x)$ *but* $\alpha(a_1)(x) \neq \alpha(a_2)(x)$, and let $\varepsilon = \|\alpha(a_1)(x) - \alpha(a_2)(x)\| > 0$. Choose a neighbourhood N of x such that $\|a_1(y) - a_2(y)\| < \varepsilon$ for $y \in N$, and let $\rho: X \to [0,1]$ be a continuous function such that $\rho(x) = 1$ and $\rho = 0$ outsi ρ de N; then $\|\rho a_1 - \rho a_2\| < \varepsilon$ and, since α is isometric,

$\|\alpha(\rho a_1)(x) - \alpha(\rho a_2)\| < \varepsilon$. *Now* $\alpha(\rho a_i)(x) = (\rho\alpha(a_i))(x) = \rho(x)\alpha(a_i)(x) = \alpha(a_i)(x)$, so that

$$\varepsilon\|\alpha(a_1)(x) - \alpha(a_2)(x)\| = \|\alpha(\rho a_1)(x) - \alpha(\rho a_2)\| < \varepsilon,$$

which is nonsense, and we have proved (i). The second part now follows if we define $\alpha_Y(b)$ for $b \in \Gamma(E|_Y)$ to be $\alpha(a)|_Y$ for any $a \in \Gamma_0(E)$ which extends b.

Lemma(1.2.2)[21]: Let E be a continuous field of C^*-algebras over a locally compact space X, let $A = \Gamma_0(E)$ and let $m \in M(A)$.

(i) If $a_1, a_2 \in A$ satisfy $a_1(x) = a_2(x)$, then $ma_1(x) = ma_2(x)$.

(ii) If $Y \subset X$ is compact, then m induces a multiplier m_Y of $\Gamma(E|_Y)$.

This can be proved along the lines of the preceding lemma, or deduced from the results of [22].

A special case of a continuous field of C^*-algebras is the trivial field $E = X \times B$, where we take for $\Gamma(E)$ the set of all continuous functions from X to B.

Lemma (1.2.3)[21]: Let X be a compact space, let B be a C^*-algebra and $\alpha \in \mathrm{Aut}_{c(X)}C(X, B)$; for $x \in X$ *we* define $\alpha_x: B \to B$ by $\alpha_x(\underline{b}) = \alpha(b)(x)$, where $\underline{b}$ is the constant function; with value b. Then $\alpha_x \in \mathrm{Aut}\, B$, the map $\alpha \to \alpha_x$ is a homomorphism, and $x \to \alpha_x$ is a continuous map of X into Aut B when Aut B has the topology of pointwise convergence. Further, if $f \in C(X, B)$ then

$$\alpha(f)(x) = \alpha_x(f(x)) \quad \text{for } x \in X.$$

Proof: The last statement follows from Lemma (1.2.1); the rest are straightforward and are the content of [52, Lemmas 3.6-3.9].

Recall that an elementary C^*-algebra is one which is isomorphic to the algebra K{H) for some Hilbert space H. Let X be a locally compact space and E be a continuous field of elementary C^*-algebras over X.

We say that E is locally trivial if it is locally isomorphic to the field $X \times K(H)$ for some Hilbert space H; we observe that these are the fibre bundles over X with fibre K{H) and structure group *Aut* K(H) (in the topology of pointwise convergence). We say that E satisfies Fell's condition if for each $x \in X$ there of E whose values are rank one projections in a neighbourhood of x.

If E is a continuous field of elementary C^*-algebras over a locally compact space X, and if E satisfies Fell's condition, then $A = \Gamma_0(E)$ is called a continuous trace C^*-algebra. The spectrum of A can be identified with X, and the primitive ideals have the form $I_x = \{a \in \Gamma_0(E): a(x) = 0\}$; E is called the field associated with A, is unique up to isomorphism, and can be recovered from A by taking $E_x = A/I_x$, see [32]. If A is a separable continuous trace C^*-algebra, then its spectrum is paracompact; frequently we shall assume that our C^*-algebras are separable. We observe that if each of the irreducible representations of a separable continuous trace C^*-algebra A has Hilbert dimension $\varkappa_0$, and if $\hat{A}$ has finite dimension, then the field associated with A is locally trivial [32, 10.8.8].

If A is a C^*-algebra such that every irreducible representation of A has dimension $n(< \infty)$, A is called an n-homogeneous C^*-algebra. Fell (see [53, Section 2]) shows that n-homogeneous C^*-algebras all have the form $\Gamma_0(E)$ for some (locally trivial) bundle of n $\times$

n matrix algebras; in particular, they are continuous trace C^*-algebras. If an n-homogeneous C^*-algebra has an identity then its spectrum X is compact; algebraists refer to these either as Azumaya algebras over $C(X)$, or as central separable $C(X)$-algebras. Conversely, it is not hard to deduce from [57] that a unital continuous trace C^*-algebra is just the finite direct sum of n-homogeneous C^*-algebras, where n can vary from summand to summand.

Lemma(1.2.4)[21]: Let A be a continuous trace C^*-algebra with spectrum X and let $\alpha \in \mathrm{Aut}\, A$. Then α is a $C_b(X)$-automorphism if and only if $\alpha(I) \subset I$ for every primitive ideal I of A.

Proof: Every primitive ideal of A has the form $I_x = \{a: a(x) = 0\}$ for some $x \in X$, and Lemma (1.2.1) tells us that $\alpha(I_x) \subset I_x$ for every $\alpha \in \mathrm{Aut}_{C_b(X)}A$. Conversely, suppose $a(I_x) \subset I_x$ for all x (i.e. $a(x) = 0 \Longrightarrow \alpha(a)(x) = 0$) and let $a \in A, f \in C_b(X)$. Let $x \in X$ then $(f(x)a)(x) = (fa)(x)$ and so $f(x)\alpha(a)(x) = \alpha(f(x)a)(x) = \alpha(fa)(x)$, as required.

Corollary(1.2.5)[21]: Let A be a continuous trace C^*-algebra with spectrum X. Then $\mathrm{Aut}_{C_B(X)}A = \Pi(A)$.

Proof: Lemma(1.2.4) implies that $\alpha(I) = I$ for every closed two-sided ideal I of A, and the result now follows from [36].

If A and B are C^*-algebras, we denote by $A \odot B$ their algebrai tensor product. If A is represented faithfully on H and B is represented faithfully on K, then $A \odot B$ is represented faithfully on $H \otimes K$, , and its closure in $B(H \otimes K)$ is a C^*-algebra $A \otimes_* B$ which is independent of the representations chosen. A C^*-algebra A is nuclear if there is a unique C^*-tensor product norm $A \odot B$ for any C^*-algebra B; in particular, continuous trace C^*-algebras are nuclear (see e.g., [55]). If A or B is nuclear then we write $A \otimes B$ for the unique C^*-tensor product of A and B.

Let A and B be C^*-algebras, and let $\pi: M\{A) \to B(H)$ and $\rho: M(B) \to B(K)$ be faithful representations; then $\pi \otimes \rho$ is a faithful representation of $A \odot B$. Thus we have an embedding of $A \otimes_* B$ as $A \odot B \subset M(A) \otimes_* M(B)$, and since $A \odot B$ is an ideal in $M(A) \otimes_* M(B)$ so is $A \otimes_* B$. The universal property of multiplier algebras implies that there is an embedding of $M(A) \otimes_* M(B)$ into $M(A \otimes_* B)$. It is straightforward that this is the obvious map: in other words,

$$\left(\sum_j m_j \otimes n_j\right)\left(\sum_i a_i \otimes b_i\right) = \sum_{ij} m_j a_i \otimes n_j b_i.$$

This embedding is not in general surjective [22, Section 3]. We shall identify $M(A) \otimes_* M(B)$ with its image in $M(A \otimes_* B)$.

A C^*-algebra ı A s said to be stable if $A \otimes K(H) \cong A$, where H is a separable infinite-dimensional Hilbert space. We recall that if E and F are continuous fields of elementary C^*-algebras satisfying Fell's condition over a space X, then we can define a field $E \otimes F$ over X whose fibre $(E \otimes F)_x$ over $x \in X$ *is* $E_x \otimes F_x$ (see [33]).

Lemma (1.2.6)[21]: Let $A = \Gamma_0(E)$ be a continuous trace C^*-algebra with par compact spectrum X. Then the map $\Phi: A \odot K(H) \to \Gamma_0(E \otimes (X \times K(H))$ defined by

$$\Phi\left(\sum_i a_i \otimes k_i\right)(x) = \sum_{ij} a_i(x) \otimes k_i$$

extends to an isomorphism of $A \otimes K(H)$ onto $\Gamma_0(E \otimes (X \times K(H)))$. The induced map $\Phi^{\wedge}: X \to (A \otimes K(H))^{\wedge}$ sends $J_x = \{b: b(x) = 0\}$ onto $I_x \otimes K(H)$, where $I_x = \{a \in A: a(x) = 0\}$.

Proof: It is obvious that Φ is injective on $A \otimes K(H)$. Hence, by uniqueness of the C^*-algebra norm, it is isometric. Since the range of Φ is clearly dense, Φ is surjective. The assertion about $\Phi^{\wedge}$ is quite easy to check.

Proposition(1.2.7)[21]: Let A be a separable continuous trace C^*-algebra with spectrum X. Then is A stable if and only if the field associated with A is locally trivial of rank $\varkappa_0$.

Proof: Suppose that A is stable, and let E be the field associated with A. Then the lemma tells us that $A \cong \Gamma_0(E \otimes (X \times K(H)))$, and it follows from [32, Section 10.5] that $E \cong E \otimes (X \times K(H))$. Since A is separable so is E, and Theoreme 2 of [33] implies that E is locally trivial; clearly each fibre has rank. $\varkappa_0$Conversely, suppose that $A = \Gamma_0(E)$ and E is locally trivial of rank $\varkappa_0$. Theoreme 1 of [33] implies that $\delta(E \otimes (X \times K(H))) = \delta(E)$, and Theoreme 2 of [33] shows that $E \otimes (X \times K(H))$ is locally trivial, so that we can deduce from [32, 10.8.4] that $E \cong E \otimes (X \times K(H))$. Thus $\Gamma_0(E) \cong \Gamma_0(E \otimes (X \times K(H)))$, and the result follows from Lemma (1.2.11).

A derivation of a C^*-algebra A is a (bounded) linear map $\delta: A \to A$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ f or $a, b \in A$; we say δ is inner if there exists $m \in M(A)$ such that $\delta(a) = ma - am$ for $a \in A$, and we write $\delta = ad\, m$. If every derivation of A is inner we write $H^1(A, M(A)) = 0$. If δ is a derivation of A then $\exp \delta$ is an automorphism, and if α is an automorphism of A close to the identity then we can define a derivation $\log \alpha$ of A by the power series expansion for log.This correspondence between automorphisms and derivations gives the following well-known result of Dixmier [58].

Proposition(1.2.8)[21]: Let A be a C^*-algebra. Then every derivation of A is inner if and only if every automorphism close to the identity has the form Ad u for some $u \in U(M(A))$ close to $1 \in M\{A).L$

Lemma(1.2.9)[21]: Let A be a C^*-algebra, and let $\alpha \in Aut\, A$. If is α close to the identity then $\alpha \in Aut_{C_{b(\hat{A})}} A$.

Proof: First we note that by the Dauns-Hofmann theorem, $C_b(\hat{A})$is the centre of the multiplier algebra M{A) of A. If $\alpha \in Aut$ A is close to the identity, then $\alpha = exp\, \delta$for some derivation δ of A. Now δ extends to a derivation $\bar{\delta}$of M(A) (via $\bar{\delta}(m)a = \delta(ma) - m\delta(a)$ etc.), and a calculation shows that $\bar{\delta}: Z(M(A)) \to Z(M(A))$; thus by [51, 4.1.2] $\bar{\delta} \equiv 0$ on Z(M(A)). Thus $\bar{\alpha} = exp\, \bar{\delta}$fixes Z(M(A)); but $\bar{\alpha}$ is an extension of α and the result follows.

We shall also need some elementary sheaf cohomology; a good reference for our purposes is chapter 5 of [55]—particularly on Čech cohomology. Let X be a paracompact space, and let $\mathfrak{R}$ and$\mathcal{Y}$ respectively denote the sheaves of germs of continuous $R-$ and $S^1 -$ valued functions on X. Then the covering map $t \to 2\pi it: R \to S^1$ induces a short exact sequence of sheaves

$$0 \to Z \to \mathfrak{R} \to \mathcal{Y} \to 0,$$

which in turn induces a long exact sequence of cohomology:

$$\ldots \to H^P(X, \mathfrak{R}) \to H^p\{X, \mathcal{Y}) \to H^{P+1}(X, Z) \to H^{P+1}(X, \mathfrak{R}) \to \ldots.$$

Since $\mathfrak{R}$ is a fine sheaf, $H^P(X, \mathfrak{R}) = 0$ for $p \geq 1$ and so we have isomorphisms $H^p(X, \mathcal{Y}) \cong H^{p+1}(X, Z)$ for $p \geq 1$; this gives us a concrete realisation of $H^2(X, Z)$ and $H^3(X, Z)$ in terms of cocycles with coefficients in S^1. Finally, we recall that if G is a topological group which acts transitively on a space F, and $\mathfrak{R}$ i the sheaf of germs of continuous G-valued functions on X, then $H^1(X, \mathcal{Y})$ is in one-to-one correspondence with the set of isomorphism classes of fibre bundles over X with fibre F and structure group G. Combining these last two observations gives us the well-known characterisation of $H^2(X, Z)$ as the set of isomorphism classes of complex line bundles over X.

We have the following theorem:

Theorem(1.2.10)[21]: Let A be a separable continuous trace C^*-algebra with spectrum X. Then there is an exact sequence

$$0 \to Inn\, A \to \mathrm{Aut}_{C_{b(X)}} A \overset{\eta}{\to} H^2(X, Z).$$

If A is stable, then η is surjective.

The proof of this result will be accomplished in several stages. We shall first reduce to the case where A is stable, so that the field E associated with A is locally trivial. We then associate to each $\alpha \in \mathrm{AutC}_{b(X)}A$ a 1-cocycle $\xi(\alpha)$ over X with coefficients in the sheaf $\mathcal{Y}$ of germs of continuous S^1 valued functions, and show that $\alpha \to \xi(\alpha)$ induces a homomorphism $\xi: \mathrm{AutC}_{b(X)}A \to H^1(X, \mathcal{Y})$; composing with the isomorphism $H^1(X, y) = H^2(X, Z)$ gives the homomorphism η. Our next step is to identify ker ξ: this is easy once we have Lemma (1.2.9) a concrete realization of $M(A)$ of a bundle with fibre $B(H)$. Finally we show Theorem (1.2.10) that £ is onto using a standard Zorn $'s$ lemma argument.

Once we have established Theorem (1.2.10), we look briefly at its implications in the case where A is an n-homogeneous C^*-algebra. We then explore the relationship between $\mathrm{AutC}_{b(X)}A$ and the group Aut A of all automorphisms of A; we prove that $Aut\ A/\mathrm{AutC}_{b(X)}A$ can be identified with a group of homeomorphisms of X Theorem (1.2.20). We close by recasting, in terms of automorphisms, the proof of the theorem of Akemann, Elliott, Pedersen and Tomiyama [3] that all derivations of separable continuous trace C^*-algebras are implemented by multipliers.

Aut K(H) will have the topology of pointwise convergence and U(H) the strong operator topology; we notice that both become topological groups. Further, it is easy to see that the strong and $*-$strong operator topologies coincide on U(H).

Let A be a separable continuous trace C^*-algebra with spectrum X.
Then $A \otimes K(H)$ is a stable separable continuous trace C^*-algebra with spectrum X, and any automorphism α of A induces an automorphism $\alpha \otimes id$ of $A \odot K(H)$; since there is a unique C^*-tensor product norm on $A \odot K(H)$, $\alpha \otimes id$ is isometric and so extends to an automorphism of $A \otimes K(H)$.
It follows from Lemma (1.2.4) and the last statement of Lemma (1.2.11) that if $\alpha \in \mathrm{AutC}_{b(X)}A$ then $\alpha \otimes id \in \mathrm{AutC}_{b(X)}A \otimes K(H)$. We recall that there is an embedding of

$M(A) \otimes_* M(K(H))$ into $M(A \otimes K(H))$; it is clear that if α is implemented by a unitary $U \in M(A)$, then $\alpha \otimes id$ is implemented by $u \otimes 1 \in M(A \otimes K(H))$. We claim that the *map* $\alpha \to \alpha \otimes id$ induces an injection

$$\mathrm{Aut}C_{b(X)}A\,/\mathrm{Inn}\,A \to \mathrm{Aut}C_{b(X)}A \otimes K(H)/\mathrm{Inn}\,A \otimes K(H).$$

Thus if we can prove Theorem (1.2.10) for stable A, then the result for general A will follow immediately. This claim is a consequence of the following lemma:

Lemma(1.2.11)[21]: If $\alpha \in \mathrm{Aut}\,A$ and $\alpha \otimes \mathrm{id}$ is an inner automorphism of $A \otimes K(H)$, then α is inner.

Proof: Suppose that $\alpha \in Aut\,A$ and $\alpha \otimes id = Ad\,u$ for a unitary $u \in M(A \otimes K(H))$. Let $e \in K(H)$ be a minimal projection; then $a \to a \otimes e$ induces an isomorphism of A with $A \otimes e \subset A \otimes K(H)$, and hence an isomorphism of $M(A)$ with $M(A) \otimes e \subset M(A) \otimes_* K(H) \subset M(A \otimes K(H))$. It is routine to check that

$$a \otimes e \to (1 \otimes e)\ u\ (a \otimes e)(1 \otimes e), \quad a \otimes e \to (1 \otimes e)(a \otimes e)\ u\ (1 \otimes e)$$

defines a multiplier of $A \otimes e$; thus there is $a\ v \otimes e \in M(A) \otimes e$ such that

$$(1 \otimes e)\ u\ (a \otimes e)(1 \otimes e) = (v \otimes e)(a \otimes e) \quad \text{for } a \in A.$$

Then a computation shows that for $a \in A$

$$\alpha(a) \otimes e = (1 \otimes e)u(a \otimes e)u^*(1 \otimes e) = vav^* \otimes e,$$

so that $\alpha(a) = vav^*$ and α is inner.

We now prove that if X is $a$compact space then $C(X)$ −module automorphisms of $C(X, K(H))$ are locally inner. Recall that if $\xi \in H$, $\|\xi\| = 1$ and p is the rank one projection of H onto $C\xi$, then the map $kp \to kp(\xi)$ is an isometric isomorphism of $K(H)p$ onto H[32, 10.6]; further, under this map $k \in K(H)$ corresponds to left multiplication by k on $K(H)p$. We shall need the following well-known lemma:

Lemma(1.2.12)[21]: If $\phi \in \mathrm{Aut}\,K(H)$, p is a rank one projection and $v \in K(H)$ satisfies $vv^* = \phi(p), v^*v = p$, then $u(hp) = \phi(h)v$ defines a unitary operator $u: K(H)p \to K(H)p$ such that $\phi = \mathrm{Ad}\,u$.

Proof: It is easy to check that u is well-defined,and a computation using the inner product $(hp|kp) = \mathrm{tr}(pk^*hp)$ shows that $u^*(kp) = \phi^{-1}(k)\phi^{-1}(v^*)$. It is now straightforward to verify that u is a nitary implementing ϕ.

Proposition(1.2.13)[21]: Let $p \in K(H)$ be a rank one projection. Then there is a continuous map $\gamma: M = \{\phi \in \mathrm{Aut}\,K(H): \|\phi(p) - p\| < 1\} \to U(H)$ such that $Ad^\circ\gamma$ is the identity on M. Further, if $\|\phi - \mathrm{id}\| < \varepsilon \le {}^1/_2$ then $\|\gamma(\phi) - 1\| < 4\varepsilon$.

Proof: Suppose that $\phi \in M$; then $\phi(p)p \neq 0$, and $v(\phi) = \phi(p)p/\|\phi(p)p\|$ defines a continuous map of M into $K(H)$. Then $v(\phi)^* v(\phi)$ is a positive element of $pK(H)p$ of norm one, and so $v(\phi)^* v(\phi) = p$; similarly $v(\phi)v(\phi)^* = \phi(P)$ thus by the lemma

$$\gamma(\phi)(hP) = \phi(h)v(\phi) \quad (hp \in K(H).P, \qquad \phi \in M)$$

defines a unitary operator $\gamma(\phi) \in U(H)$and $\phi = \mathrm{Ad}\,\gamma(\phi)$. It is easy to verify that γ is continuous, and a computation shows that if $\|\phi - \mathrm{id}\| < \varepsilon \le 1/2$, then$\|v(\phi) - p\| < 3\varepsilon$ and $\|\gamma(\phi) - 1\| < 4\varepsilon$, which completes the proof.

Corollary(1.2.14)[21]: Let X be a compact space, and let $\alpha \in \mathrm{Aut}_{C(X)} C(X, K(H))$. Then for each $x_0 \in X$ there is aneighbourhood N of x_0 and a(strong operator) continuous map $u: N \to U\{H)$ such that

$$\alpha(f)(x) = u(x)f(x)u(x)^* \quad \text{for } x \in N, f \in C(X, K(H)).$$

Let A be a separable stable continuous trace C^* −algebra with spectrum X, and let $\alpha \in \mathrm{Aut}_{C(X)} A$. Then there is a locally trivial field E of elementary C^*-algebras of rank $\aleph_0$ over x such that $A = \Gamma_0(E)$.

Let $\{M_j\}$ be an open cover of X such that there are isomorphisms $h_j . \bar{M}_j \times K(H) \to E|_{\bar{M}_j}$ then, αinduces automorphisms (by Lemma (1.2.1))

$$\alpha_j = (h_j^{-1})_* \circ \alpha_{\bar{M}_j} \circ (h_j)_* \in \mathrm{Aut}_{C(X)} C\left(\bar{M}_j, k(H)\right)$$

(without loss of generality we have assumed that each $\bar{M}_j$. is compact). According to Corollary (1.2.14) we can by shrinking the M_j's assume that there are continuous maps $u_j: \bar{M}_j j \to U(H)$ such that u_j. implements α_j.If we start the argument of [32, 10.7.11] with this open cover, we obtain:

Proposition(1.2.15)[21]: Let $A = \Gamma_0(E)$ be a separable stable continuous traceC^* −algebra with spectrum X, and let $\alpha \in \mathrm{Aut}_{C(X)} A$. Then there is an open cover $\{N_i\}$ of X and

(i) isomorphisms $h_i: \bar{N}_i \times K(H) \longrightarrow E|_{\bar{N}_i}$;

(ii) continuous maps $v_{ij}: . \bar{N}_{ij} \longrightarrow U(H)$ such that

$$(h_j^{-1})_x (h_j)_x = \mathrm{Ad}\, v_{ij}\,(x) \quad \text{for} x \in \bar{N}_{ij};$$

(iii) continuous maps $u_i: . \bar{N}_i \longrightarrow U(H)$ such that $\alpha_i = \mathrm{Ad}\, u_i$.

We now observe that for $x \in \bar{N}_{ij}$ (using the notation of the proposition) we have

$$(\alpha_i)_* = (h_i^{-1})_x \circ \alpha_x \circ (h_i)_x$$
$$= (h_i^{-1} h_i)_x \circ (h_i^{-1})_x \circ \alpha_x \circ (h_i)_x \circ ((h_i^{-1} h_i)^{-1})_x,$$

so that on $\bar{N}_{ij}$

$$Ad\, u_i = Ad\, v_{ij} \circ Ad\, u_j \circ Ad\, v_{ij}^* = Ad\,(v_{ij} u_j v_{ij}^*).$$

Two unitaries in $B(H)$ can induce the same automorphism of $K(H)$ only if they differ by a constant of modulus 1; hence there are continuous maps $\lambda_{ij}\ \bar{N}_{ij} \to S^1$such that

$$\lambda_{ij}(x) u_i(x) = \left(v_{ij} x\right) u_j(x) \left(v_{ij} x\right)^*, \quad \text{for } x \in \bar{N}_{ij}.$$

Further, it follows from (2) that $Ad\,(v_{ij} v_{ik}) = Ad\,(v_{ik})$ so that on $\bar{N}_{ijk}$ we have

$$\lambda_{ij}\, \lambda_{ik} u_i = v_{ij} [V_{IK} U_K V_{IK}^*]\, v_{ij}^* = V_{IK} U_K V_{IK}^*,$$

so that $\lambda_{ij} \lambda_{ik} = \lambda_{ik}$ and $\{N_i, \lambda_{ij}\}$ defines a 1-cocycle with coefficients in the sheaf y of germs of S^1 −valued functions. We shall denote by $\xi(\alpha)$ the class in $H^1(X, y)$ defined by $\{N_i, \lambda_{ij}\}$.

Our next task is to show that $\xi(\alpha)$ depends only on α and not on any of the choices we have made. We first note that if we replace our cover by a finer one, and the h_i'S, v_{ij}'S and u_i'S and hence also the λ_{ij}'S —by their restrictions, this will not change the cohomology class (see for example [55, page 201]), so we suppose that $\{N_i\}$ is an open cover of X and that we have

(i) isomorphisms $h_i, g_i : \overline{N}_i . \times K(H) \to E|_{\overline{N}_i}$;

(ii)continuous maps $v_{ij}, w_{ij} : \overline{N}_{ij} \to U(H)$ such that $(h_i^{-1}h_j)_* = \mathrm{Ad}\, v_{ij}$ and $(g_i^{-1}g_j)_* = \mathrm{Ad}\, w_{ij}$ on $\overline{N}_{ij}$;

(iii)continuous maps $u_i, s_i : \overline{N}_i \to U(H)$ such that on $\overline{N}_i, (h_i^{-1})_* \circ \alpha_{\overline{N}_i} \circ (h_i)_* = \mathrm{Ad}\, u_i$ and$(g_i^{-1})_* \circ \alpha_{\overline{N}_i} \circ (g_i)_* = \mathrm{Ad}\, s_i$.

The corresponding cocycles $\{\lambda_{ij}\}$ and $\{\mu_{ij}\}$ are given by

$$\lambda_{ij}1 = v_{IJ}U_JV_{IJ}^*U_I^* \quad \text{and} \quad \mu_{ij}1 = W_{IJ}S_JW_{IJ}^*S_I^* \;;$$

to show these define the same class in $H^1(X, \mathcal{Y})$ we shall construct maps $v_1 : \overline{N}_i \to S^1$ such that $\mu_{ij} = v_i^{-1}.\lambda_{ij}V_j$ *on* $\overline{N}_{ij}$.

By Corollary (1.2.14) we can assume (shrinking the $\overline{N}_i$'S if necessary) that there are continuous maps $z_1 : \overline{N}_i \to U(H)$ such that $(g_i^{-1}, h_i)_* = Ad\, z_i$ Then (ii) and (iii) imply that

$$Ad\,(z_i^* w_{ij}Z_i) = Ad\, v_{ij} \;\; and \;\; Ad\, s_i = Ad\,(z_iu_iz_i^*);$$

from this last identity we deduce that there are continuous maps $v_i : \overline{N}_i \to S^1$ with $s_i = v_iz_iu_iz_i^*$ on $\overline{N}_i$ We now compute:

$$\mu_{ij}1 = w_{ij}\big(v_jz_ju_jz_j^*\big)w_{ij}^*(v_i^{-1}z_iu_i^*z_i^*) = v_i^{-1}v_jz_i\big(z_i^*w_{ij}z_j\big)u_j\big(z_i^*w_{ij}^*z_j\big)u_i^*z_i^*$$
$$= v_i^{-1}v_jz_iv_{ij}u_iv_{ij}v_{ij}^*u_iz_i^* = v_i^{-1}v_j\lambda_{ij}1\,,$$

which shows that $\{\lambda_{ij}\}$ and $\{\mu_{ij}\}$ represent the same class in $H^1(X, y)$. We conclude that $\xi(\alpha)$ is well-defined.

If $\alpha, \beta \in \mathrm{Aui}_{C_b(X)}A$, then we can proceed and construct data for and β satisfying the conclusion of Proposition (1.2.15) with respect to the same cover of X; if $\alpha_1 = \mathrm{Ad}\, u_i$ and $\beta_i = Ad\, s_i$. then $(\alpha \circ \beta)_i = Ad\,(u_is_i)$ and it is a straightforward calculation to check that the resulting cocycle for aβ is the product of the cocycles for α and β. Since $\xi\,(\alpha\beta\,)$ is independent of all the choices made, it follows that $\xi : \mathrm{Aut}_{C_b(X)}A \to H^1(X, \mathcal{Y})$ is a group homomorphism.

We next have to identify the kernel of ξ as the inner automorphisms; since ξ is defined in terms of local coordinates we must first identify multipliers in terms of local coordinates. We shall need the following well-known lemma; it is the trivial case of [22, Corollary 3.5], and a simple proof can also be given using Lemma (1.2.2) and the fact that $M(K(H)) = B(H)$.

Lemma(1.2.16)[21]: Let X be a compact space. Then every multiplier of $C(X, K(H))$ is given b a $*-$strong continuous map of X into B(H).

Let $A = \Gamma_0(E), \{N_i\}, h_i$ and v_{ij} be as in Proposition (1.2.15). If m is a multiplier of A, then by Lemma (1.2.2) m induces multipliers m_i of $C\big(\overline{N}_i, K(H)\big)$ defined by $m_I' = (h_i^{-1})_* \circ m_{\overline{N}_i}' \circ (h_i)^*$, and similarly for m_i''. By Lemma (1.2.16) each m_i. is given by a *-

strongly continuous function $t_i: \overline{N}_i \to B(H)$; then for $x \in \overline{N}_{ij}$ and $b \in C(\overline{N}_{ij}, K(H))$ we have

$$t_i(x)b(x) = (h_i^{-1})_x(m_x(h_i)_x(b(x)))$$
$$= \left(h_i^{-1}h_j\right)_x \left[(h_i^{-1})_x\left(m_x(h_i)_x\left\{\left(h_i^{-1}h_j\right)_x b(x)\right\}\right)\right]$$
$$v_{ij} = (x)\,[t_j(x)\, v_{ij}(x)^*(b\{x)v_{ij}(x)]\, v_{ij}(x)^*$$

so that $t_i = v_{ij}t_jV_{ij}^*$. Conversely, it is routine to check that any collection of *-strong continuous maps $t_i: \overline{N}_i \to B(H)$ satisfying $t_i = v_{ij}t_jv_{ij}^*$ on $\overline{N}_{ij}$ defines a multiplier m of $\Gamma_0(E)$ by $m'(a)(x) = (h_i)_x[t_i(x)(h_i^{-1})_x(a(x))], m''(a)(x) = (h_i)_x\,(h_i^{-1})_x(a(x)t_i(x)]$. Thus we have

Proposition(1.2.17)[21]: Let $A = \Gamma_0(E)$ be a separable stable continuous trace C^*- algebra with spectrum X, and let N_i, h_i, v_{ij} satisfy conditions(I)and(II)of Proposition (1.2.15)**.**Then we can identify M(A) with the set $\{\{t_i\}: t_i: N_i \to B(H)$ *is* *-strong continuous, uniformly bounded and $t_i = v_{ij}t_jv_{ij}^*$ on $N_{ij}\}$.

Let $A = \Gamma_0(E)$be a separable stable continuous trace C^* –algebra with spectrum X, and let N_i, h_i. and v_{ij} satisfy conditions (i) and (ii) of Proposition (1.2.15). If $\alpha = Ad\, u \in$ Inn A, then by the preceding proposition we can regard u as a family of maps $u_i: \overline{N}_i \to U(H)$ satisfying $u_i =..\, v_{ij}u_jv_{ij}^*$.

The usual sort of calculation shows that u_i implements α_i, and the analogous λ_{ij}*'s* are all 1 so that $\xi(\alpha) = 0 \in H^1(X, \mathcal{Y})$. Now suppose that $\alpha \in \mathrm{Ant}_{C_b(X)}A$ and $\xi(\alpha) = 0$, so that there are N_i, h_i, v_{ij} and u_i satisfying the conclusions of Proposition (1.2.33) and continuous maps $v_i: \overline{N}_i \to S^1$ such that on $\overline{N}_{ij}$

$$v_i^{-1}v_ju_i = \lambda_{ij}u_i = v_{ij}u_jv_{ij}^*.$$

If we define $w_i: \overline{N}_i \to U(H)$ by $w_i = v_i^{-1}u_i$, then $\alpha_i = Ad\, w_i$ and it is easy to check that $w_i = v_{ij}w_jv_{ij}^*$. Thus the w_i*'s* define a unitary element w of M(A) and another calculation shows that $\alpha = Ad\,(w)$**, so** that $\alpha \in Inn\, A$. This completes the proof that ker $\xi = Inn\, A$.

To complete the proof of Theorem (1.2.10) we have to show that ξ is surjective. Our method is a version of Lance*'s* argument [45, last part of Theorem 4.3]; modified (cf. [32, proof of 10.8.4]) to allow for transition functions and noncompact spectra. We begin with a simple lemma.

Lemma(1.2.18)[21]: Let A be a separable stable continuous trace C^* –algebra with spectrum X, and let N_i, h_i. and v_{ij} satisfy (I) and (II) of Proposition (1.2.15). If $u_i: \overline{N}_i \to U(H)$ are continuous maps such that $Ad\,(v_{ij}u_jv_{ij}^*) = Ad\, u_i$ on $\overline{N}_{ij}$, then there is a unique $C_b(X)$-module automorphism α of A such that $\alpha_i = Ad\, u_i$..

Proof: Let $a \in A$, and let

$$\alpha(a)(x) = (h_i)_x \circ \operatorname{Ad} u_i(x) \circ (h_i^{-1})_x(a(x)) \text{for } x \in \overline{N}_{ij}.$$

Then a(a) is a well-defined element of A, and it is clear that α is a a C_b (X)-algebra homomorphism; a is an automorphism since we can write down its inverse. Using a partition of unity we can see that a C_b (X)-automorphism is uniquely determined by its restrictions to theN_i's , and hence α is unique.

Let $A = \Gamma_0(E), N_i, h_i.$ and v_{ij} be as in (i) and (ii) of Proposition (1.2.15), and let $\{N_i, \lambda_{ij}\}$ be a 1-cocycle with coefficients in$\mathcal{Y}$. Since $A$is separable, X is paracompact and so we can assume that the cover $\{N_i\}_{i\in I}$ is locally finite; moreover, by refining, we can assume that $\{\overline{N}_i\}_{i\in I}$is a locally finite cover. We shall show that there are maps $u_i: N_i \to U(H)$ satisfying $v_{ij}\, u_j v_{ij}^* = \lambda_{ij}$on N_{ij} *9* so that by Lemma (1.2.18) the u_i's will define an automorphism $\alpha \in Autc_{b(X)}\ with\ \xi(\alpha) = \{N_i, \lambda_{ij}\}$. Let $\mathfrak{A}$ denote the sheaf of germs of continuous U(H)-wahxed functions on X we shall need the fact that $\mathfrak{A}$ is soft, which follows by Lemma 4.2 of [45] from the contractibility of U(H) [32].

Let $\mathcal{Y} = \{(J, \beta): J \subseteq 1$ and $\beta = \{\beta_i\}_{i\in J}$ consist of β_i of $\mathfrak{A}$ over $\overline{N}_i$ such that $v_{ij}\beta_j v_{ij}^* = \lambda_{ij}\beta_i$. on $\overline{N}_i \cap N_j\}$. If $J \subseteq K$ and $\beta_i = \gamma_i$ for $i \in J$, then we set $(J, \beta) \leq (K, \gamma)$: this is a partial order on $\mathcal{Y}$The collection $\mathcal{Y}$ is non-empty and every chain has an upper bound, so by Zorn's lemma $\mathcal{Y}$ contains a maximal element (J, β).Suppose that there is an $i \in I\backslash J$, and let $R = \overline{N}_i \cap \left(\bigcup_{j\in J} \overline{N}_j\right)$; note that R is closed since $\overline{N}_i$ is locally finite. Suppose that $j, k \in J$, *so* that $v_{ij}\beta_k v_{ik}^* = \lambda_{ik}\beta_j$ on $\overline{N}_j \cap \overline{N}_k$. Then on $\overline{N}_i \cap \overline{N}_j \cap \overline{N}_k$

$$\lambda_{ij}^{-1} v_{ij}\beta_j v_{ij}^* = \lambda_{ij}^{-1} v_{ij}\lambda_{ik}^{-1} v_{jk}\beta_k v_{jk}^* v_{ij}^* = \lambda_{ik}^{-1} v_{ik}\beta_k v_{ik}^*$$

so that $\beta_i = \lambda_{ij}^{-1} v_{ij}\beta_j v_{ij}^*$ defines a continuous β_i of $\mathcal{U}$ over R, which satisfies the right relations on $R \cap \overline{N}_j$. for all $j \in J$. Since $\mathcal{U}$ is soft, β_i extends to a continuous β_i over $\overline{N}_i$, which contradicts the maximality of (J, β) so that J must be all of I. If we restrict the β_i's to N_i., then they are continuous of $\mathcal{U}$ over open sets and so given by continuous maps $u_i: N_i \to U(H)$ which have the required properties. This completes the proof of Theorem (1.2.10).

We now investigate the special case of the above construction where A is an n-homogeneous C^*-algebra, or, in algebraic language, an Azumaya algebra. Let A be an algebra identity over a commutative ring R, and let M be a left $A \otimes_R A^{\circ P}$module—that is, M is a left and a right **A**-module and the action of R commutes with everything. We say M is invertible if there is another left $A \otimes_R A^{\circ P}$-module N with $M \otimes_A N \cong A$ and $N \otimes_A M \cong A$, and we denote by $\operatorname{Pic}_R A$ the group of isomorphism classes of invertible left $A \otimes_R A^{\circ P}$-modules. If for $\alpha \in Aut_R A$ we define ${}_\alpha A_1$ to be the $A \otimes_R A^{\circ P}$-module with A as underlying set, and left and right multiplication defined by $a . b = \alpha(a)b$ and $b. a = ba$ respectively, then ${}_\alpha A_1$ is invertible and the $map\ \alpha \to {}_\alpha A_l$ induces an antihomomorphism π such that

$$(*) \qquad 0 \to Inn\, A \to Aut_R\, A \xrightarrow{\pi} Pic_R A$$

is an exact sequence (cf. [26, page 73-74]). If A is an Azumaya R-algebra (that is, the centre of A is R and A is a projective $A \otimes_R A^{\circ P}$-module) then every $A \otimes_R A^{\circ P}$-module M is isomorphic to $A \otimes_R Z(M)$, where $Z(M) = \{m \in M: am = ma \text{ for all } a\}$. [24, Theorem 3.1]. The correspondence $P \to A \otimes_R P$ induces an isomorphism $Pic\, R \cong Pic_R A$, where

Pic R $= Pic_R R$ denotes the usual Picard group of invertible R-modules. Thus for an Azumaya algebra A over R we have the exact sequence

$$0 \rightarrow Inn\,\mathrm{A} \rightarrow Aut_R\, A \overset{\rho}{\rightarrow} Pic\,\mathrm{R},$$

where for $\alpha \in Aut_R\, A, \rho(\alpha)$ is represented by

$$J_\alpha = \{\mathrm{a} \in \mathrm{A}: \alpha(\mathrm{b})\mathrm{a} = \mathrm{ab} \quad \text{for all b} \in \mathrm{A}\}.$$

This result is due to Rosenberg and Zelinsky ([50]; cf. also [31]); Knus [44] has shown that the range of p is contained in the torsion subgroup of Pic R.

Let A be Azumaya algebra over $\mathrm{C(X)}$ for a compact space X. First of all, it follows from [23] that modulo the inners the $C(X)$-automorphisms and the $C(X)$-algebra automorphisms coincide, so that we can write $Aut_{\mathrm{c(X)}}A$ without causing any confusion. The natural equivalence $\mathrm{E} \rightarrow \Gamma(\mathrm{E})$ between vector bundles over X and projective $C(X)$-modules allow us to interpret Pic C(X) as $\mathrm{H}^1(\mathrm{X}, \mathcal{Y}) \cong \mathrm{H}^2(\mathrm{X}, \mathrm{Z})$, and it's not hard to see that under this identification our homomorphism η of Theorem (1.2.10) and the Rosenberg-Zelinsky homomorphism p coincide. Knus's result tells us that if $\mathrm{H}^2(\mathrm{X}, \mathrm{Z})$ is torsion free then every $C(X)$-automorphism of every n-homogeneous C^* −algebra with spectrum X is inner. When Kadison and Ringrose in [42, Section 4, example (d)] were looking for a space X for which $\pi(\mathrm{C}(\mathrm{X}, \mathrm{M_n}(\mathrm{C}))) \neq \mathrm{Inn}\,\mathrm{C}(\mathrm{X}, \mathrm{M_n}(\mathrm{C}))$, they took for X the projective unitary group $\mathrm{U(n)}/\mathrm{S}^1$; it turns out that $\mathrm{H}^2(\mathrm{U(n)}/\mathrm{S}^1; \mathrm{Z}) \cong \mathrm{Z_n}$ [27, Section 4]. Notice that they had to choose different spaces X for different fibre dimensions.

Recently Brown, Green and Rieffel [28] have introduced a C^*-version of the Picard group, which we denote by Pic^*:if A is a C^*-algebra then Pic^*A consists of equivalence classes of $A - A -$ imprimitivity bimodules (cf.[49, Definition 6.10]). Their Pic^*A corresponds to the algebraic $\mathrm{Pic_c A}$, and the appropriate generalisation of the exact sequence (*) of the preceding is

$$0 \rightarrow Inn\,\mathrm{A} \rightarrow Aut\, A \overset{\pi}{\rightarrow} Pic^*\,\mathrm{A};$$

this is Proposition (1.2.41) of [28]. Brown, Green and Rieffel prove that if A is stable and has a strictly positive element (for example, if A is separable and stable) then the antihomomorphism π if we surjective [28, Corollary 3.5]. denote by $\mathrm{Pic}^*_{\mathrm{ZM(A)}}\mathrm{A}$ the subgroup of $\mathrm{Pic}^*\mathrm{A}$ consisting of (classes of) **A-A**-imprimitivity bimodules X such that $\mathrm{ar.x.b} = \mathrm{a.x.rb}$ for all $\mathrm{x} \in \mathrm{X}, \mathrm{a}, \mathrm{b} \in \mathrm{A}$ and $\mathrm{r} \in \mathrm{ZM(A)}$ then it is routine that $\pi(\alpha) \in \mathrm{Pic}^*_{\mathrm{ZM(A)}}\mathrm{A}$ if and only if $\alpha \in \mathrm{Aut_{ZM(A)}A}$. In particular, for a separable stable continuous trace $\mathrm{C}^* -$ algebra we obtain an isomorphism $\mathrm{Aut_{ZM(A)}A/Inn} \cong \mathrm{Pic}^*_{\mathrm{ZM(A)}}\mathrm{A}$, which together with Theorem (1.2.10) shows that $\mathrm{H}^2(\mathrm{X}, \mathrm{Z}) \cong \mathrm{Pic}^*_{\mathrm{C_B(X)}}\mathrm{A}$. We do not know how to prove this result directly, although such a proof would be of interest; it would also be interesting to find out for what class of C^* −algebras we do have $\mathrm{H}^2(\widehat{\mathrm{A}}, \mathrm{Z}) \cong \mathrm{Pic}^*_{\mathrm{ZM(A)}}\mathrm{A}$. Such a result together with [28, Corollary 3.5] would of course give our Theorem (1.2.10), but we observe that this approach will not give the results of since the algebras we consider there are not stable, so that Corollary 3.5 of [28] does not apply.

Let A be a separable continuous trace C^* −algebra with spectrum X, and let $\delta(\mathrm{A})$ be the class in $\mathrm{H}^3(\mathrm{X}, \mathrm{Z})$ associated to A by Dixmier and Douady [34], [32, Section 10.7]; their construction goes as follows. If $\{\mathrm{N_i}\}, \mathrm{h_i}$ and $\mathrm{v_{ij}}$ satisfy the conditions (i) and (ii) of Proposition (1.2.15), then $\mathrm{Ad}\,(\mathrm{v_{ij}v_{jk}}) = \mathrm{Ad}\,\mathrm{v_{ik}}$ on $\mathrm{N_{ijk}}$, and so there are continuous maps

t_{ijk}: $N_{ijk} \rightarrow S^1$ such that $v_{ij}v_{ik} = t_{ijk}v_{ik}$. A completely routine calculation (see [32, proof of 10.7.12]) shows that $\{N_i, t_{ijk}\}$ is a 2-cocycle with coefficients in $\mathcal{Y}$ and so determines an element of $H^2(X, \mathcal{Y})$. This cohomology class depends only on A [32, 10.7.12] and is denoted by $\gamma(A)$; its canonical image in $H^3(X, Z) \cong H^2(X, \mathcal{Y})$ is denoted by $\delta(A)$. Now let $\alpha \in$ Aut A. Then α induces a homeomorphism $\tilde{\alpha}$ of X by sending the primitive ideal $I_x = \{a \in A: a(x) = 0\}$ to $\alpha(I_x) = I_{\tilde{\alpha}(x)}$ and $\tilde{\alpha}$ in turn induces an automorphism $\tilde{a}^*$ of $H^3(X, Z)$. In fact $\tilde{\alpha}$ fixes the Dixmier-Douady class $\delta(A)$. We denote by $\mathrm{Hom}_{\delta(A)}X$ the set of homeomorphisms of X with this property. Observe that the homeomorphism $\tilde{\alpha}$ is not the natural map $\hat{\alpha}: \hat{A} \rightarrow \hat{A}$ induced by α; our $\tilde{\alpha} = (\alpha^{-1})\hat{}$. We have adopted this notation so that the map a $\rightarrow \tilde{a}$ / is a homomorphism and not an anti-homomorphism.

Theorem (1.2.19)[21]: Let A be separable continuous trace C^* −algebra with spectrum X. Then there is an exact sequence

$$0 \rightarrow \mathrm{Aut}_{C_b(x)}A \rightarrow \mathrm{Aut}\, A \overset{\rho}{\rightarrow} \mathrm{Hom}_{\delta(4)}X.$$

If A is stable, then ρ is surjective.

Proof: If $\alpha \in$ Aut A we set $\rho(\alpha) = \tilde{\alpha}: X \rightarrow X$; it is easy to check that p is a group homomorphism, and Lemma (1.2.4) tells us that ker ρ is $\mathrm{Aut}_{C_{b(x)}}$ A. It remains to verify that $\rho(\alpha)^*$ fixed $\delta(A)$ and that ρ is onto if A is stable. We first show that $\rho(\alpha)^*(\delta(A)) = \delta(A)$; again we reduce to the case where A is stable by passing to $\alpha \otimes \mathrm{id} \in \mathrm{Aut}\, A \otimes K(H)$. As before if we identify the spectra of A and $A \otimes K(H)$ via the correspondence $I \rightarrow I \otimes K(H)$ then it is routine to verify that $\rho(\alpha) = \rho(\alpha \otimes \mathrm{id})$ and that $a \in \mathrm{Aut}_{C_{b(x)}}A$ exactly when $\alpha \otimes \mathrm{id} \in \mathrm{Aut}_{C_{b(x)}})A \otimes K(H)$. Further, by [33] and Lemma (1.2.11) we have that $\delta(A) = \delta(A \otimes K(H))$ so that $\rho(\alpha)$ fixes $\rho(A)$ if and only if $\rho(\alpha \otimes \mathrm{id})$ fixes $\delta(A \otimes K(H))$.

So we suppose that A is stable and $\alpha \in$ Aut A. Let $\{N_i\}_{i \in I}, h_i$ and v_{ij} be as in (i) and (ii) of Proposition (1.2.15), so that$\gamma(A)$ is represented by $\{N_i, t_{ijk})$ where $v_{ij}v_{jk} = t_{ijk}v_{ik}$. Under the homeomorphism a this cocycle is carried to $\{\tilde{\alpha}^{-1}(N_i), t_{ijk} \circ \tilde{\alpha}\}$ and we must show that this also represents $\gamma(A)$. We observe that if $A = \Gamma_0(E)$ then the automorphism a induces isomorphisms $\alpha_x: E_x \rightarrow E_{\tilde{\alpha}(x)}$ and if $Y \subset X$ is compact, then

$$\alpha_Y(f)(\tilde{\alpha}(x)) = \alpha_x(f(x)) \quad (x \in Y)$$

defines an isomorphism $\alpha_Y: \Gamma(E|_Y) \rightarrow \Gamma(E|_{\tilde{a}(Y)})$ (cf. the argument of Lemma 1.2.4). We define

$$g_i: \tilde{\alpha}^{-1}(\overline{N}_i) \times K(H) \rightarrow E|_{\tilde{\alpha}^{-1}(\overline{N}_i)} \text{by } (g_i)_y = (\alpha^{-1})_{\tilde{\alpha}(Y)} \circ (h_i)_{\tilde{\alpha}(Y)};$$

the preceding observation shows that the g_i's are isomorphisms of fields. If we alsodefine $w_{ij}: \tilde{\alpha}^{-1}(\overline{N}_{ij}) \rightarrow U(H)$ by $w_{ij} = v_{ij} \circ \tilde{\alpha}$, then a calculation gives

$$\mathrm{Ad}\, w_{ij}(y) = (g_i^{-1}g_j)_y \quad \text{for } y \in \tilde{\alpha}^{-1}(\overline{N}_{ij}),$$

so that the cover $\{\tilde{\alpha}^{-1}(N_i)\}$ and the g_i's, w_{ij}'s also satisfy (i) and (ii) of Proposition (1.2.15). Thus if we set $s_{ijk}w_{ik} = w_{ij}w_{ik}$ then the cocycle $\tilde{\alpha}^{-1}(N_i), s_{ijk})$ also represents the class $\gamma(A)$ in $H^2(X, \mathcal{Y})$ by [32, 10.7.12 (iii)]. But another calculation shows that $s_{ijk} = t_{ijk} \circ \tilde{\alpha}$, so we have proved that $\tilde{\alpha}^*(\gamma(A)) = \gamma(A)$ and hence that $\rho(\alpha) \in \mathrm{Hom}_{\delta(A)}X$.

Finally, we prove that ρ is surjective if A is stable. Let $\phi \in \mathrm{Hom}_{\delta(A)}X$, so that $\phi^*$and $(\phi^{-1})^*: H^3(X,Z)\ H^3(X,Z)$ both fix $\delta(A)$. If $\mathcal{Y}$ denotes the sheaf of germs of Aut K(H)-valuied functions, then Dixmier and Douady defined a map $\Delta: H^1(X,\mathcal{Y}) \to H^2(X,\mathcal{Y})$ and proved that it is bijective [34,Lemma 22]; if $\{N_i, \mathrm{Ad}\, v_{ij}\}$ as above determine $c \in H^1(X,\mathcal{Y})$, then $\Delta(c)$ is represented by $\{N_i, t_{ijk}\}$ and so equals $\gamma(A)$. It is clear the maps $(\phi^{-1})^*: H^1(X,\mathcal{Y}) \to H^1(X,\mathcal{Y}$and $(\phi^{-1})^*: H^1(X,\mathcal{Y}) \to H^1(X,\mathcal{Y})$commute with Δ; hence if $(\phi^{-1})^*$ fixes $\delta(A)$ then $\{N_1 \mathrm{Ad}\, v_{ij}\}$ and $\{\emptyset(N_i)9\ \mathrm{Ad}\, v_{ij} \circ \phi^{-1}\}$ define the same class in $H^1(X,\mathcal{Y})$. This means that there is a common refinement $\{Mp\}_{p\in P}$ of the covers $\{N_i\}$ and $\{\phi(N_i)\}$, functions $\tau, \sigma: P \to 1$such that $M_p \subset N_{\tau(p)}$ and $M_p \subset \phi(N_{\sigma(p)})$ and continuous maps $\beta_P: M_p \to \mathrm{Aut}\, K(H)$ such that

$$\mathrm{Adv}_{\tau(p)\tau(q)} = \beta_p \circ (\mathrm{Adv}_{\sigma(p)\sigma(q)} \circ \phi^{-1}) \circ \beta_q^{-1} \text{ on} M_{pq}.$$

We can regard elements of A variously as collections of maps $a_i: N_i \to K(H)$ satisfying $a_i = \mathrm{Ad}\, v_{ij} a_j$on N_{ij} or as collections $a_p: M_p \to K(H)$ satisfying $a_p = \mathrm{Ad}\, v_{\tau(p)\tau(q)} a_q$ on M_{pq}. For $a \in A$ we define $\alpha(a) \in A$ by $\alpha(a)_p = \beta_p(a_{\sigma(p)} \circ \phi^{-1})$ **the** calculation

$$\mathrm{Adv}_{\tau(p)\tau(q)}(\alpha(a)_q) = (\mathrm{Adv}_{\tau(p)\tau(q)})\beta_q(a_{\sigma(p)} \circ \phi^{-1}) = \beta_p[\mathrm{Adv}_{\tau(p)\tau(q)} \circ \phi^{-1}](a_{\sigma(p)} \circ \phi^{-1})$$
$$= \beta_p(a_{\sigma(p)} \circ \phi^{-1}) = \alpha(a)_p$$

shows that $\alpha(a)$ is a well-defined element of A. In fact α is an automorphism of A, and it remains to check that $\rho(\alpha) = \phi$ To do this we need to show that $a(x) = 0$ implies $\alpha(a)(\phi(x)) = 0$; but this follows at once from the observation that $\alpha(a)(y) = 0$ for $y \in M_p$ if and only if $\alpha(a)_p(y) = 0$.

We conclude by proving the following theorem of Akemann et al. [3]. The proof is only a minor modification of their proof; however, it will show how to prove a similar result for other C^*-algebras which arise of bundles.

Theorem(1.2.20)[21]: Let A be a separable continuous trace C^*-algebra. Then every derivation of A is inner.

Proof: Let $A = \Gamma_0(E)$ have spectrum X and let δ be a self-adjoint derivation of A. Let $\alpha_t = \exp t\delta$ so that for small t $||\alpha_t - \mathrm{id}||$ is small. Now, by 1.15 α_t, is in $\mathrm{Aui}_{C_b(X)}A$ and so α_t is locally inner. By taking logs one sees that δ is "locally inner" i.e., there is an open locally finite cover $\{N_i\}$ of X and elements $\{x_i\}$ in M{A) so that for a in A which is supported on N_i, $\delta(a) = x_i a - a x_i$ If $\{\rho_i\}$ is a partition of unity subordinate to $\{N_i\}$, then $x = \sum \rho_i x_i$ is in M(A) and $\delta = \mathrm{ad}\, x$.

We shall now generalise Lance's theorem in two directions: we shall allow locally trivial bundles in place of the trivial bundle $X \times B(H)$, and we shall vary the fibre algebra. Throughout this X will be a separable compact space and B will be a C^*-algebra with identity; the groups Inn B and U(B) will have their respective norm topologies. We shall be concerned with the space $A = \Gamma(E)$ of a bundle E over X with fibre B and structure group Inn B.

By Lemma (1.2.1) an automorphism $\alpha \in \mathrm{Aut}_{c(X)}A$ induces automorphisms of the fibres E_x; if each of these is inner we say α is pointwise inner, and we denote by Pinn A the group of such automorphisms. We shall give conditions on the fibre B which imply that $\mathrm{PInn}\, A/\mathrm{Inn}\, A \cong H^2(X,Z)$. The crucial step in our argument is to show that (under conditions on B) a pointwise inner automorphism α of $C(X, B)$ is locally inner in the sense that the unitaries implementing the fibre automorphisms α_x can be chosen continuously near each point. We observe that both Lance [45] and Smith [52] recognised this as a major step in their analysis of Aut $C(X, B(H))$. The approach has two ingredients: a theorem of Kallman and Elliott, and a simple selection theorem argument. As a corollary of this we see that in many cases PInn A $= \pi(A)$, so that the result is a true generalisation of Lance's. We also observe that these conditions on the fibre B ensure that $H^1(A, A) = 0$. We conclude by observing that the Dixmier-Douady classification of bundles of C^* −algebras by third Čech cohomology also works in our setting.

The following result is due in this generality to Elliott, although the first theorems along these lines were proved by Kallman. This is from [37], but we refer to [38] for further details.

Theorem (1.2.21). Let B be the quotient of an AW^* −-algebra by a closed two-sided ideal, and let ϕ_n be a sequence of automorphisms of B such that $\|\phi_n b - b\| \to 0$ for each $b \in B$. Then $\|\phi_n - id\| \to 0$.

Corollary(1.2.22)[21]: Let B be a quotient of an AW^*-algebra, X be a separable compact space and $\alpha \in Aut_{c(x)}C(X, B)$. Then the induced map $x \to \alpha_x : X \to Aut\, B$ of Lemma (1.2.3) is continuous when Aut B has the norm topology.

Proposition (1.2.23)[21]: Let B be a C^* −algebra with identity 1 such that $H^1(B, B) = 0$. Then, there is a continuous map$\gamma: \{\alpha \in Aut\, B: \|\alpha - id\| < \sqrt{3}\} \to U(B)$ such that $\alpha = Ad\, \gamma\, (\alpha)$.

Proof: Let D_* be the closed real linear subspace of L(B) consisting of self-adjoint derivations and let B_* be the closed real linear subspace of B consisting of skew-adjoint elements. Since $H^1(B, B) = 0$ we have that $ad: B_* \to D_*$ is a surjection and so by the Bartle-Graves selection theorem [59] there is a continuous map $: g\, D_* \to *$ such that $ad\,(g(d)) = d$ for all $d \in D_*$ and $(g_0) = 0$. Let $\alpha = exp \circ g \circ log$, then the argument of [58, III. 9.4] applies.

Theorem(1.2.24)[21]: Let X be a separable compact space, and let B be a C^* −algebra with identity satisfying

(i) $H^1(B, B) = 0$;

(ii) B is the quotient of an AW^* −algebra by a closed two-sided ideal.

Then every pointwise inner automorphism of $C(X, B)$ *is* locally inner.

Proof: Let $\alpha \in Aut_{C(X)}C(X, B)$ be pointwise inner, so that the map $x \to a_x$ of Lemma (1.2.3). takes values in Inn B. By Corollary (1.2.22), $x \to \alpha_x$ is continuous, and the result now follows from Proposition (1.2.23), see [48].

We now turn to the more general situation where the C^*-algebras are spaces of bundles. The first result shows that our pointwise inner automorphisms coincide in many cases with the **iT**-inner automorphisms of Kadison and Ringrose.

Proposition (1.2.25)[21]: Let B be a von Neumann algebra,let E be a bundle over X with fibre B and structure group Inn B and let $A = \Gamma(E)$. Then the pointwise inner automorphisms of A are precisely the π-inner automorphisms of A.

Proof: Suppose first that α is π-inner. Since $Z\phi((A)) \subseteq Z(\overline{(\phi A)})$ for any representation ϕ of A (where $\phi\overline{(A)}$ denotes the weak closure of ϕA)), it follows immediately that α is a $C(X)$-automorphism and so induces automorphisms $\alpha_x$of the fibres E_x. If B acts faithfully as a von Neumann algebra on the Hilbert space H, then we can use the representation $a \to \sum_x^{\oplus} a(x)$ on $\sum_x^{\oplus} H$ to see that α is pointwise inner.

Conversely, suppose that α is pointwise inner and $\phi: A \to B(H)$ is a faithful representation such that $\phi(1) = 1$. We can choose an open cover $\{N_i\}_{i=1}^n$ of X such that $E|_{\overline{N}_i}$ is trivial, and by Theorem (1.2.24) we can assume (byshrinking the N_i 's if necessary) that the induced automorphisms α_i of $C(\overline{N}_i, B)$ are inner. Consider the finite increasing sequence $0 = I_0 \subset I_1 \subset \ldots \subset I_n = A$ of ideals of A corresponding to the open sets $\phi, N_l \cup N_1, \cup N_2, \ldots, N_1 \cup \ldots \cup N_n = X$. For each $K = 1, \ldots, n, I_{k-1}$ and I_k are fixed by α and the induced automorphism of I_k/I_{k-1} is inner. It follows immediately that α a is π-inner (cf. [36]).

Corollary(1.2.26)[21]: If B is a von Neumann algebra, then the π-inner automor-phisms of $C(X, B)$ are precisely the locally inner automorphisms.

Proof: This is a combination of Theorem (1.2.24). and Proposition (1.2.25) for the trivial bundle $X \times B$.

Let B be a C^*-algebra with identity satisfying the hypotheses of Theorem (1.2.24), and let E be a bundle over X with fibre B and structure group Inn B. Then the transition functions of E form a cocycle $\{N_i, \phi_{ij}: 1 \leq \imath \leq n\}$ with coefficients in the sheaf of germs of Inn B-valued functions. Using Proposition (1.2.23) and a covering argument like that of [32,10.7.11] we can refine the cover $\{N_i\}$ so that the maps ϕ_{ij}have the form Ad v_{ij} for continuous maps $v_{ij}: N_{ij} \to U(B)$. If α is a point wise inner automorphism of A, then by Theorem (1.2.24) we can shrink the $N_i's$ again so that the automorphisms induced by α on $C(\overline{N}_i, B)$ are all inner. Provided the centre of B is trivial, the arguments of go through in this case, and there is a homomorphism $\eta: \text{PInn } A \to H^1(X, S)$. We obtain the following theorem:

Theorem(1.2.27)[21]: Let B bea C^*-algebra with identity such that

(i) $Z(B) = C1$;

(ii) $H^1(B, B) = 0$;

(iii) B is the quotient of an A W^*-algebra by a closed two-sided ideal.

Let X be a separable compact space, let E be a bundle over X with fibre B and structure group Inn B, and let $A = \Gamma(E)$. Then there is an exact sequence

$$0 \to Inn\, A \to PInn\, A \xrightarrow{\eta} H^2(X, Z).$$

If in additionU(B)iscontractible,then η is surjective.

We now consider the case where the fibres do not have trivial centre. Again let B be a C^*-algebra with identity satisfying (i) and (ii) of Theorem (1.2.24), and let E be a bundle over X with fibre B and group Inn B; we write $A = \Gamma(E)$. The same construction associates to each $\alpha \in PInn\, A$ a 1-cocycle with coefficients in the sheaf $\mathcal{Y}$ of $U(Z(B))$-valued functions. Now Z(B)is a commutative $C^* -$ algebra, and hence isomorphic to $C(Y)$ for the compact space $Y = Z(B)^\frown$, under this isomorphism $U(Z(B))$ is carried to the group $C(Y, S^1)$ of continuous functions from Y to the circle S^1. Let $\mathscr{Y}$ denote the sheaf of germs of $C(y, R)$-valued functions; then the map $f \rightarrow exp\ 2\pi if$ induces a short exact sequence of sheaves:

$$0 \rightarrow C(Y, Z) \rightarrow \mathscr{Y} \rightarrow \mathfrak{T} \rightarrow 0.$$

The sheaf $\mathscr{Y}$ is fine, and so the corresponding exact sequence of cohomology implies that $H^1(X, \mathfrak{T}) = H^2(X, C(Y, Z))$. The group $C(Y, Z)$ is just $H^\circ(Y, Z)$ (it is easily seen that $C(Y, Z)$ is discrete as a subset of $C(Y, R)$) and our theorem becomes.

Theorem (1.2.28)[21]: Let B be a C^*-algebra with identity such that
(i) $H^1(B, B) = 0$;
(ii) B is a quotient of an AW^* −algebra.

Let E be a bundle over a separable compact space X with fibre B and structure group InnB, and let $A = \Gamma(E)$. There is an exact sequence

$$0 \rightarrow \text{Inn}, A \rightarrow \text{PInn } A \overset{\eta}{\rightarrow} H^2(X, H^\circ(Z(B)^\frown, Z));$$

if U(B) is contractible, then η is surjective.

As we have already noted, hypotheses (i) and (ii) are automatically satisfied if B is an AW^*-algebra. The question of contractibility of U(B) though, is an interesting one. Kuiper's famous theorem asserts that U(H) is contractible, and Breuer [29] has extended this to show that for any properly infinite semifinite countably decomposable von Neumann algebra B the group U(B) is contractible (in particular, if B is a type 1_∞ or 11_∞ algebra acting on a separable space H). Araki, Smith and Smith [23] and Handelman [40] have shown that this is not the case for von Neumann algebras of type 11_1 by computing $\pi_1(U\{B))$. It has been conjectured that U(B) will always be contractible if B is properly infinite.

For the algebra $A = C(X, B(H))$ Lance proved that the group Inn A coincides with the connected component γ (A) of the identity in Aut A. That InnA ⊂ γ(A) is an immediate consequence of Kuiper's theorem, and so this also holds for $A = C(X, B)$ whenever U(B) is contractible. The converse inclusion will be true for C(X, B) whenever the hypotheses of Theorem (1.2.24) hold for B. Putting this observation together with Proposition (1.2.25) and Theorem (1.2.28) gives the following direct generalisation of Lance's main theorem [45,Theorem 4.3]:

Theorem(1.2.29)[21]: Let X be a separable compact space, let B be a properly infinite semifinite countably decomposable von Neumann algebra, and let Y be the spectrum of the centre of B. Then

$$\pi(C(X, B)) / \gamma(C(x, B) \cong H^2(X, H^\circ(Y, Z)).$$

We conclude by observing that the Dixmier-Douady classification of locally trivial bundles of elementary C^*-algebras also works for the bundles we have been considering.

Proposition (1.2.30)[21]: Let B be a C^* −algebra with identity and suppose that
(i) Ad: U(B) → Inn B is a fibre bundle;
(ii) U(B) is contractible.

Then for each paracompact space X there *is* a one-to-one correspondence between $H^3(X, H^\circ(Z(B)^\frown, Z))$ and the set of isomorphism classes of bundles over X with fibre B and structure group Inn B.

Proof: Let $\mathscr{g}, \mathfrak{A}$ respectively denote the sheaves of germs of Inn B and $U(B)$-valued functions. Then isomorphism classes of bundles over X with fibre B and group Inn B correspond to the cohomology classes in $H^1(X, \mathscr{g})$. The fibre of the bundle $U(B) \to$ Inn B is the set $U(B) \cap Z(B)$, which can be identified with $C(Z, (B)^\frown, S^1)$. If we denote by $\mathfrak{T}$ the sheaf of germs of $C(Z, (B)^\frown, S^1)$-valued functions, then the fact that $U(B) \to$ Inn B is a bundle says there is a short exact sequence of sheaves

$$0 \to \mathfrak{T} \to \mathfrak{A} \to \mathscr{g} \to 0$$

Since $U(B)$ is contractible, $\mathfrak{A}$ is soft and there is a bijection of $H^1(X, \mathscr{g})$ onto $H^2(X, \mathfrak{T})$ [34]. $H^2(X, \mathfrak{T}) \cong H^3(X, C(Z(B)^\frown, Z)) = H^3(X, H^\circ(Z(B)^\frown, Z))$ and we're done.

Corollary (1.2.31)[370]: Let $p^r \in K(H)$ be a rank one projection. Then there is a continuous map $\gamma_r: M = \{\phi^r \in \text{Aut}\, K(H): \|\phi^r(p^r) - p^r\| < 1\} \to U(H)$ such that $Ad^\circ\gamma$ is the identity on M. Further, if $\sum_r \|\phi^r - id_r\| < \varepsilon \le \frac{1}{2}$ then $\sum_r \|\gamma_r(\phi^r) - 1\| < 4\varepsilon$.

Proof: Suppose that $\phi^r \in M$; then $\sum_r \phi^r(p^r)p^r \neq 0$, and $\sum_r v(\phi^r) = \sum_r \phi^r(p^r)p^r / \|\phi^r(p^r)p^r\|$ defines a continuous map of M into K(H). Then $v(\phi^r)^* v(\phi^r)$ is a positive element of $p^r K(H) p^r$ of norm one, and so $\sum_r v(\phi^r)^* v(\phi^r) = \sum_r p^r$; similarly $\sum_r v(\phi^r) v(\phi^r)^* = \sum_r \phi^r(p^r)$ thus by the lemma

$$\sum_r \gamma_r(\phi^r)(hp^r) = \sum_r \phi^r(h) v(\phi^r) \quad (hp^r \in K(H).p^r, \qquad \phi^r \in M)$$

defines a unitary operator $\gamma_r(\phi^r) \in U(H)$ and $\phi^r = \sum_r Ad_r\, \gamma_r(\phi^r)$. It is easy to verify that γ is continuous, and a computation shows that if $\sum_r \|\phi^r - id_r\| < \varepsilon \le 1/2$, then $\sum_r \|v(\phi^r) - p^r\| < 3\varepsilon$ and $\sum_r \|\gamma_r(\phi^r) - 1\| < 4\varepsilon$, which completes the proof.

Chapter 2

Automorphisms and Countable Degree-1 Saturation

We introduce notions of metric ω_1-trees and coherent families of Polish spaces and develop their theory parallel to the classical theory of trees of height ω_1 and coherent families indexed by a σ-directed ordering. We present unified proofs of several properties of the corona of σ-unital C*-algebras such as AA-CRISP,SAW^*, being sub-σ-Stonean in the sense of Kirchberg, and the conclusion of Kasparov's Technical Theorem. We obtain results about the quotient of these Banach algebras by their ideal of compact operators being C^* algebras which hve the countable degree -1 saturation propertyin the model theory sense of I. We also obtain results about quasicentral approximate units, multipliers and duality.

Section (2.1): Automorphisms of all Calkin Algebras

For an infinite-dimensional complex Hilbert space H. Let $\mathcal{B}(H)$ be its algebra of bounded linear operators, $\mathcal{K}(H)$its ideal of compact operators and $C(H) = \mathcal{B}(H)/\mathcal{K}(H)$the Calkin algebra. Answering a question first asked by Brown-Douglas-Fillmore, in [109] and [104] it was proved that the existence of outer automorphisms of the Calkin algebra associated with a separable H is independent from ZFC. We consider the existence of outer automorphisms of the Calkin algebra associated with an arbitrary complex, infinite-dimensional Hilbert space.

PFA stands for the Proper Forcing Axiom, MA for Martin's Axiom and TA stands for Todorcevic's Axiom (see e.g., [111] or [107] for PFA and TA and [106] for MA). It is well-known that both MA and TA are consequences of PFA.

Theorem (2.1.1)[99]: TA implies all automorphisms of the Calkin algebra on a separable, infinite-dimensional Hilbert space are inner.

All of these results are part of the program of finding set-theoretic rigidity results for algebraic quotient structures. This program can be traced back to Shelah's seminal construction of a model of ZFC in which all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ are trivial ([110]). At present we have a non-unified collection of results and it is unclear how far-reaching this phenomenon is (see [101], [102], [103] and [104]).

The idea of the proofs of Theorem (2.1.11) and Theorem (2.1.23) is taken from the analogous Velickovic's results on automorphisms of the Boolean algebra $\mathcal{P}(\mathcal{K})/\mathrm{Fin}$ in [112].

If Φ is an automorphism of $\mathcal{P}(\omega_1)/\mathrm{Fin}$ then there is a closed unbounded set $C \subseteq \omega_1$ such that for every $\alpha \in \mathrm{C}$ the restriction of Φ to $\mathcal{P}(\alpha)/\mathrm{Fin}$ is an automorphism of $\mathcal{P}(\alpha)/\mathrm{Fin}$. Since MA and TA imply that all automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ are trivial ([13]), for each $\alpha \in C$ we can fix a map $h_\alpha\colon \alpha \to \alpha$ such that the map $\mathcal{P}(\alpha) \ni A \to h_\alpha[A] \in \mathcal{P}(\alpha)$ is a representation of the restriction of $\Phi$to $\mathcal{P}(\alpha)/\mathrm{Fin}$.

For $\alpha < \beta < \gamma$ with β and γ in C we have that $h_\beta \restriction \alpha$ and $h_\gamma \restriction \alpha$ agree modulo finite. Therefore

$$T = \{h_\beta \restriction \alpha : \alpha < \beta, \beta \in C\},$$

considered as a tree with respect to the extension ordering, has countable levels. Automorphism $\Phi$is trivial if and only if T has a cofinal branch. For every $f\colon \omega_1 \to 2$ the tree

$$T[f] = \{f \circ t : t \in T\}$$

has a cofinal branch, determined by Y such that $[Y]_{Fin} = \Phi([X]_{Fin})$, where $f = \chi x$. On the other hand, if f is added by forcing with finite conditions $\mathbb{P}$ (i.e., if $\dot{f}$ codes a set of $\aleph_1$ side-by-side Cohen reals over V) then $\mathbb{P}$ forces that $T[\dot{f}]$ has no cofinal branches. Applying MA to the poset for adding $\dot{f}$ followed by the ccc poset for specializing $T\ [\dot{f}]$ one obtains a contradiction.

Velickovic's proof of triviality of automorphisms of $\mathcal{P}(k)$/Fin for $k \geq \aleph_2$ uses a PFA-reflection argument, in which the above proof is preceded by a Levy collapse of k to $\aleph_1$.

While the structure of our proof of Theorem (2.1.11) loosely resembles the above sketch, a number of nontrivial additions and modifications were required. For example, it is not clear whether for every automorphism Φ of $C(\ell_2(\aleph_1))$ the set C of countable ordinals α such that the restriction of Φ to $C(\ell_2(\alpha))$ is an automorphism of the latter algebra is closed and unbounded. This follows from $MA + TA$ by Theorem (2.1.11), but I don't know whether this fact is true in ZFC. This problem is dealt. An another inconvenience was caused by the fact that the natural 'quantized' analogue of the poset for adding $\aleph_1$ Cohen reals is not ccc (Lemma 2.1.14), as well as the expected non-commutativity complications.

Also, the appropriate analogues of Velickovic's trees T and $T[f]$ are continuous rather than discrete. Therefore the proof of Theorem (2.1.11) required introduction and analysis of 'metric w_1-trees,' analogous to the classical theory of w_1-trees. It is 'purely set-theoretic' in the sense that C^*-algebras are not being mentioned in it.

Metric w_1-trees and metric coherent families are introduced and treated using MA and PFA, few simple and well-known general facts about inner automorphisms of C^*algebras. We define analogues of trees T and $T[f\ ]$ from Velickovic's proof, and we analyze $T\ [\mathcal{T}]$ for an appropriately defined generic operator proof of Theorem (2.1.23) and brief concluding remarks can be found.

The background on C^*algebras and set theory are [100] and [106], respectively. Applications of combinatorial set theory to C^*algebras can be found in [113] and [105].

We introduce a continuous version of Aronszajn trees. In operator algebras 'contraction' commonly refers to a map that is distance-non-increasing. In some other areas of mathematics such maps are referred to as 1-Lipshitz and 'contraction' refers to a distance-decreasing map. The latter type of a map is referred to as a strict contraction by operator algebraists. In what follows I use the operator-algebra ic terminology, hence a contraction f is assumed to satisfy $d(\mathcal{X},\mathcal{Y}) \geq d(f(\mathcal{X}), f(\mathcal{Y}))$. Other than this concession, the theory of operator algebras does not make appearance.

A metric ω_1-tree is a family $T = (X_\alpha, d_\alpha, \pi_{\beta a},$ for $\alpha \leq \beta < \omega_1)$, such that

(i) X_α is a complete metric space with compatible metric d_α.
(ii) $\pi_{\beta\alpha} : X_\beta - X_\alpha$ is a contractive surjection,
(iii) projections $\pi_{\beta\alpha}$ are commuting and $\pi_{\alpha\alpha} = \mathrm{id}_{X_\alpha}$ for all α.

If all spaces X_α are separable we say T is a Polish ω_1-tree. If in addition the inverse limit $\lim_{\alpha\leftarrow} X_\alpha$ is empty then we say that T is a Polish Aronszajn tree. Otherwise, the elements of the inverse $\lim_{\leftarrow a} X_\alpha$ are considered to be branches through T. All branches and all e-branches are assumed to be cofinal.

When each d_a is a discrete metric then the above definitions reduce to the usual definitions of ω_1 -trees and Aronszajn trees (see e.g., [106]). Similarly, ε-branches, ε-

antichains and e-special trees as defined below are branches, antichains, and special trees, respectively, when $0 < \varepsilon < 1$.

Spaces X_α are assumed to be disjoint and we shall identify T with the union $\bigcup_\alpha X_\alpha$ of its levels when convenient and the projections are clear. On T we have a map Lev: $T \to \omega_1$ defined by $Lev(x) = \alpha$ if and only if $x \in X_\alpha$.

It will be convenient to write π_α for the map $\bigcup_{\beta \geq \alpha} \pi_{\beta,\alpha}$ from T into T_α. Define a map ρ on T^2 as follows. For $\mathcal{X}, \mathcal{Y}$ in T let $\alpha = \min\left(Lev(\mathcal{X}), Lev(\mathcal{Y})\right)$ and let

$$\rho(\mathcal{X}, \mathcal{Y}) = d_\alpha\left(\pi_\alpha(\mathcal{X}), \pi_\alpha(\mathcal{Y})\right).$$

Note that ρ is not a metric or even a quasi-metric. The triangle inequality is violated by any triple such that $x \neq z \; but \; \mathcal{Y} = \pi_\alpha(x) = \pi_\alpha(z)$.

For $\varepsilon > 0$ a subset A of T is an ε-anticha ρ in of T if $(\mathcal{X}, \mathcal{Y}) > \varepsilon$ for all distinct $\mathcal{X}$ and $\mathcal{Y}$ in A. We say that T is e-special if there are ε-antichains A_n, for $n \in \mathbb{N}$, such that $X_\alpha \cap \bigcup_n A_n$ is dense in X_α, for all $\alpha < \omega_1$.

For $\varepsilon > 0$ a subset A of T is an ε-branch if $A = \{x_\alpha : \alpha < w_1\}, Lev(\mathcal{X}_\alpha) = \alpha$ for all a, and $\rho(x_\alpha, x_\beta) \leq \varepsilon$ for all $a, \beta. A$ subtree of T is a subset $S \subseteq$ T that is closed under projection maps and intersects every level X_α.

Lemma (2.1.2)[99]: The following are equivalent for every metric W_1-tree T and $\varepsilon > 0$.

(i) Thas an ε-branch,

(ii) There is $B \subseteq$ T that intersects cofinally many levels such that $\rho(\mathcal{X}, \mathcal{Y}) \leq \varepsilon$ for all $\mathcal{X}, \mathcal{Y}$ in B,

(iii) Thas a subtree of diameter $\leq \varepsilon$.

Proof: For $B \subseteq T$ let its downwards closure $S(B)$ be the subset of T such that its intersection with X_α is the metric closure of $\{\pi_\alpha(x) : x \in B, \alpha \leq Lev(x)\}$. Since each π_α us p-nonincreasing, the 'ρ-diameter' of $S(B)$ is equal to the 'ρ-diameter' of B. This shows that (i) implies (iii), and the other implications do not require a proof.

Lemma (2.1.3)[99]: Assume T is a metric ω_1-tree such that each of its subtrees has an e-branch for every $\varepsilon > 0$. Then T has a branch.

Proof: Choose B_n, for $n \in \mathbb{N}$, so that $B_n$is a $1/n$-branch and $B_{n+1} \subseteq S(B_n)$. Then for every α we have that $B_n \cap X_\alpha$, for $n \in \mathbb{N}$, is a decreasing sequence of subsets of X_α with diameters converging to 0. If x_α is the unique point in $\bigcap_n (B_n \cap X_\alpha)$ thenthe fact that the projections are commuting contractions easily implies that x_α, for $\alpha < \omega_1$, is a branch of T.

There is a Polish Aronszajn tree with an ε-branch for all $\varepsilon > 0$ but no branches. To see this, fix any special Aronszajn tree T. Let X_α be the disjoint union of countably many copies of the α-th level of T and define d_α so that the the n-th copy has diameter $1/n$ and the distance between two distinct copies is 1. With the natural projection maps, the n-th copy of T includes a $1/n$-branch but T has no branches.

In the following lemma and elsewhere no attempt was made to find optimal numerical estimates.

Lemma (2.1.4)[99]: If T is an ε-special metric ω_1-tree then it has no $\varepsilon/2$-branches.

Proof: Let A_n, for $n \in \mathbb{N}$, be ε-antichains with dense union in each level. Assume x_α, for $\alpha < \omega_1$, is an ε-branch. Let n be such that $d_\alpha(x_\alpha, z_\alpha) < \varepsilon/4$ for some $z_\alpha \in A_n \cap X_\alpha$for uncountably many α. Since projections are contractions, for such $\alpha < \beta$ we have $\rho(z_\alpha, z_\beta) < \varepsilon$, a contradiction.

The proof of the following lemma is a straightforward modification of the well-known analogous fact for ω_1-trees.

Lemma (2.1.5)[99]: (MA). Assume T is a Polish ω_1-tree with no ε-branches. Then T is $\varepsilon/2$-special.

Proof: For each α fix a countable dense subset Z_α of X_α. Let $\mathbb{P}_0$ be the poset of finite $\varepsilon/2$-antichains included in $\cup_\alpha Z_\alpha$ ordered with $\mathrm{p} \geq \mathrm{q}\ if\ \mathrm{p} \subseteq \mathrm{q}$.

We shall prove $\mathbb{P}_0$ is ccc. Fix $\mathrm{p}_{\boldsymbol{\alpha}}, \alpha < \omega_1$in $\mathbb{P}_0$. Since each Z_α is countable, by a Δ-system argument we can find $\bar{\alpha}$, an uncountable $J \subseteq \omega_1$, and $(writing\ Z = \bigcup_{\beta \leq \bar{\alpha}} Z_\beta)$ $\bar{\mathrm{p}} \subseteq Z$ and $\bar{\mathrm{q}} \subseteq Z$ so that the following hold for all $\alpha \in J$. First, $\mathrm{p}_\alpha = \bar{\mathrm{p}}\ U\ \mathrm{q}_{\boldsymbol{\alpha}}$. Second, $\pi_{\bar{\alpha}}$ maps $\mathrm{q}_{\boldsymbol{\alpha}}$ injectively onto $\bar{\mathrm{q}}$. Third, $\gamma(\alpha) = min\{Lev(x) : x \in \mathrm{q}_{\boldsymbol{\alpha}}\}$ converges to ω_1.

It suffices to find $\alpha < \beta$ in J such that $\mathrm{q}_\alpha\ U\ \mathrm{q}_{\boldsymbol{\beta}}$ is an $\varepsilon/2$-antichain. Let $n = |\bar{\mathrm{q}}|$ and fix an enumeration $\mathrm{q}_\alpha = \{z_a(i) : i < n\}$ for all $\alpha \in \mathrm{J}$. Let $\mathcal{U}$ be a uniform ultrafilter on J. Assuming α and β as above cannot be found, there are $i < j < n$ such that the set $J_1 = \{\alpha \in J : \{\beta : \rho(z_\alpha(i), z_\beta(j)) < \varepsilon/2\} \in \mathcal{U}\}$ belongs to $\mathcal{U}$. But then $\rho(z_\alpha(i), z_\gamma(i)) < \varepsilon$ for all $\alpha < \gamma$ in J_1, and therefore $\{z_\alpha(i) : \alpha \in J_1\}$ defines an ε-branch of T.

This proof that $\mathbb{P}_0$ is ccc shows that it is powefully ccc, i.e., the finitely supported product $\mathbb{P}_0^{<\omega}$ of countably many copies of $\mathbb{P}$ is ccc. Apply MA to the ccc poset $\mathbb{P} = \mathbb{P}_0^{<\omega}$and $\aleph_1$ many dense sets assuring that $\mathbb{P}$ ads countably many ε-antichains A_n whose union is equal to $\bigcup_{\boldsymbol{\alpha}} Z_\alpha$.

The material of this plays a role only in the proof of Theorem (2.1.23).

A system $\mathbb{F} = (X_\lambda, d_\lambda, \pi_{\lambda,\lambda}: \lambda < \lambda'\ in\ \Lambda)$ is a coherent family of Polish spaces if

(i) Λ is upwards σ-directed set and a lower semi-lattice,
(ii) X_λ is a Polish space with compatible metric d_λ,
(iii) $\pi_{\lambda,\lambda}$: $X_{\lambda'} \to X_\lambda$ is a contractive surjection,
(iv) projections $\pi_{\lambda,\lambda}$are commuting and $\pi\lambda\lambda = \mathrm{idx}_\lambda$ for all λ.

The family is trivial if $\lim\limits_{\longleftarrow\lambda} X_\lambda \neq 0$. Hence if $\Lambda = \omega_1$ with its natural ordering then $\mathbb{F}$ is a Polish ω_1-tree.

Spaces X_λ are assumed to be disjoint and we shall identify $\mathbb{F}$ with the union $\bigcup_\lambda X_\lambda$of its levels when convenient and when the choice of projections is clear from the context. On $\mathbb{F}$ we have a map Lev: $\mathbb{F} \longrightarrow \Lambda$ defined by $\mathrm{Lev}(x) = \lambda$ if and only if $x \in X_\lambda$. It will be convenient to write π_λ for the map$\bigcup_{\lambda' \geq \lambda} \pi_{\lambda'\lambda}$.

Define a map ρ on $\mathbb{F}^2$ as follows. For x, y in $\mathbb{F}$ let $\lambda = Lev(x) \wedge Lev(y)$ and let

$$\rho(x, y) = d_\lambda(\pi_\lambda(x), \pi_\lambda(y)).$$

For $\varepsilon > 0$ a subset A of T is an ε-antichain of $\mathbb{T}$ if $\rho(x, y) > \varepsilon$ for all distinct x and y in A. A set $\{x_\lambda : \lambda \in \Lambda\}$ is an ε-branch of $\mathbb{F}$ if $x_\lambda \in X_\lambda$ for all λ and $p(x_\lambda, x_{\lambda'}) \leq \varepsilon$ for all λ and λ'.

If $Y_\lambda \subseteq X_\lambda$ is a nonempty Polish subspace for all λ and the family Y_λ , for $\lambda \in \Lambda$, A G A, is closed under the projection maps then (with d'_λ denoting the restriction of d_λ to Y_λ we say that $\mathbb{F}' = (Y_\lambda, d_\lambda, \pi_{\lambda'\lambda}, for\ \lambda < \lambda'\ in\ \Lambda)$ is a cofinal subfamily of $\mathbb{F}$.
Proof of the following is analogous to the proof of Lemma (2.1.3).

Lemma (2.1.6)[99]: Assume $\mathbb{F}$ is a coherent family of Polish spaces such that each of its cofinal subfamilies has an ε-branch for every $\varepsilon > 0$. Then $\mathbb{F}$ is trivial.

Assume $\mathbb{F}$ is a coherent family of Polish spaces. If $f: \omega_1 \rightarrow \mathbb{F}$ is a strictly increasing map then we say the Polish $\omega_1 -$ tree $(X_{f(\alpha)}, d_{f(\alpha)}, \pi_{f(\beta)f(\alpha)}, \alpha \leq \beta < \omega_1)$ is a Polish sub tree of $\mathbb{F}$.

Lemma(2.1.7)[99]: (PFA). Assume $\mathbb{F} = (X_\lambda, d_\lambda, \pi_{\lambda'\lambda} : \lambda < \lambda'\ in\ \Lambda)$ is a coherent family of Polish spaces with no ε-branches. Then $\mathbb{F}$ has an $\varepsilon / 6 -$special Polish subtree.

Proof: Let $\mathbb{p}$ denote the σ-closed collapse of $|\Lambda|$ to $\aleph_1$. Then $\mathbb{p}$ forces that there is a strictly increasing, cofinal map $f: \omega_1 \longrightarrow \Lambda$. We first prove that $\mathbb{p}$ forces the Polish ω_1-tree $T_f = (X_{f(\alpha)}, d_{f(\alpha)}, \pi_{f(\beta)f(\alpha)}, \alpha \leq \beta < \omega_1)$ has no $\varepsilon/3$-branches.

Assume otherwise and let B be a name for an $\varepsilon/3$-branch of T_f. Let $\theta = (2^{|\Lambda|})^+$ and let M be a countable elementary submodel of H_θ containing $\mathbb{F}, \mathbb{P}$, and a name $\dot{f}$ for f. Let D_n, for $n \in \mathbb{N}$, enumerate all dense open subsets of $\mathbb{P}$ that belong to M. Pick conditions $P_s, \mathcal{X}_s$ and $\mathcal{Y}_s$ for $s \in 2^{<N}$, satisfying the following for all s.

(i) $P_s \geq P_t\ if\ t$ extends s,
(ii) $P_s \in M \cap D_n$, where $n = |s|$,
(iii) $P_s \Vdash \tilde{x}_s \in B$,
(iv)$x_s \in M$, and
(vi)$\rho(x_{S0}, x_{S1}) \geq \varepsilon$.

These objects are chosen by recursion. If P_s has been chosen, then the set $\{x \in \mathbb{F} : (\exists q \leq P_s) q \Vdash x \in \dot{B}\}$ is not an ε-branch and therefore we can choose x_{so} and x_{s1} in this set such that $\rho(x_{s0}, x_{s1}) \geq \varepsilon$. Let P_{s0} and P_{s1} be (necessarily incompatible) extensions of P_s forcing that x_{so} and x_{s1}, respectively, belong to $\dot{B}$. Since all the relevant parameters are in, , P_{so} , P_{s1} , x_{so} and x_{s1} can also be chosen to belong to M .

Since Λ is σ-directed, let $\lambda(M) \in \Lambda$ be an upper bound for $M \cap \Lambda$. For each $g \in 2^{\mathbb{N}}$ let p_g be $(M, \mathbb{P})$-generic condition extending all $p_{g|n}$ and deciding $x_g \in X_{\lambda(M)}$ in $\dot{B}$. For $g \neq g'$ let s be the longest common initial segment of g and g'. We may assume g extends $s0$ and g' extends $s1$. Let $\alpha = \min(Lev(x_{s0}), Lev(x_{s1}))$ and let $\mathcal{Y}_0, \mathcal{Y}_1, x_0, x_1$ be the projections of x_g, x_g , x_{s0} and x_{s1}, respectively, to X_α. Then

$$d_\alpha(\mathcal{Y}_0, \mathcal{Y}_1) \geq d_a(x_0, x_1) - d_a(\mathcal{Y}_0, x_0) - d_a(\mathcal{Y}_1, x_1) \geq \varepsilon/3,$$

and therefore $d_{\lambda(M)}(x_g, x_{g'}) \geq \varepsilon/3$. This contradicts the assumed separability of $X_{\lambda(M)}$

Since $\mathbb{P}$ forces that $\mathbb{F}$ has no $\varepsilon/3$-branches, by Lemma (2.1.5) we have a $\mathbb{P}$-name for a ccc poset that $\varepsilon / 6 -$specializes T_f. By applying PFA to the iteration and an appropriate collection of dense sets we obtain the desired conclusion.

Coherent families of discrete Polish spaces and their uniformization using PFA have been used. See e.g., [111] and [107].

We state and show some well-known results about inner automorphisms of C^* −algebras. Recall that for a partial isometry $v$in algebra Ac by Ad v we denote the conjugation map Ad $v(a) = vav^*$.

Lemma (2.1.8)[99]: Assume that unitaries v and w in a $C^* - a$lgebra A are such that $Ad\ v$ and Adw agree on A. Then vw* ∈ Z(A).

Proof: We have $vav^* = waw^*$ and therefore $w^*va = aw^*v$ for all $a \in A$.

In the following $\dot{a}$ denotes the image of $a \in \mathcal{B}(H)$ in the Calkin algebra under the quotient map, not a forcing name.

Lemma(2.1.9)[99]: If v and w in $\mathcal{B}(H)$ are such that v and w are unitaries in C(H) and $(Ad\ v)a - (Ad\ w)a$ is compact for all $a \in \mathcal{B}(H)$, then there is $z \in \mathbb{T}$ such that $v — zw$ is compact.

Proof: We first check (a well-known fact) that $Z(C(H)) = \mathbb{C}$. Since it is a C^* −algebra, it suffices to see that the only self-adjoint elements of $Z(C(H))$ are scalar multiples of the identity. Assume $\dot{a}$ is self-adjoint and its essential spectrum is not a singleton, say it contains some $\lambda_1 < \lambda_2$. Fix $\varepsilon < |\lambda_1 — \lambda_2|/3$. In $\mathcal{B}(H)$ fix infinite -dimensional projections p and q such that $\|pap — \lambda_1 p\| < \varepsilon$ and $\|qaq — \lambda_2 q\| < \varepsilon$. A noncompact partial isometry v such that $vv^* \leq p$ and $v^*v \leq q$ clearly does not commute with a modulo the compacts.

By Lemma (2.1.8) applied to $\dot{v}$ and $\dot{w}$ and the above there is a scalar z such that $z\dot{v} = \dot{w}$, as required.

Lemma(2.1.10)[99]: Assume H is an infinite-dimensional Hilbert space and Φ and Ψ are automorphisms of $C(H)$ that agree on the corner $\dot{p}C(H)\dot{p}$ for every projection $p \in \mathcal{B}(H)$ *with* separable range. Then $\Phi = \Psi$.

Proof: We may assume H is nonseparable. Assume the contrary and let $a \in \mathcal{B}(H)$ be such that $\dot{b} = \Phi(\dot{a}) — \Psi(\dot{a}) \neq 0$. Let r be a projection with separable range such that rbr is not compact and let p be such that $\Phi(\dot{p}) = \dot{r}$. By our assumption, $\Psi(\dot{p}) = \dot{r}$. Also $\dot{r}\Psi(\dot{a})\dot{r} = \Psi(\dot{p}\dot{a}\dot{p}) = \Phi(\dot{p}\dot{a}\dot{p}) = \dot{r}\Phi(\dot{a})\dot{r}$, contradicting the choice of a.

Theorem (2.1.11)[99]: MA and TA together imply all automorphisms of the Calkin algebra associated with Hilbert space with basis of cardinality$\mathcal{N}_1$ are inner.

Proof. Let H denote $\ell_2(\aleph_1)$. We assume Φ is an automorphism of $C(H)$ and $\Phi_*: \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$ is its representation, i.e., any map such that the diagram

$$\begin{array}{ccc} \mathcal{B}(H) & \xrightarrow{\Phi_*} & \mathcal{B}(H) \\ \downarrow & & \downarrow \\ \mathcal{C}(H) & \xrightarrow{\Phi} & \mathcal{C}(H) \end{array}$$

commutes. Since every projection in $C(H)$ lifts to a projection in $\mathcal{B}(H)$ ([113]) we may assume Φ_* maps projections to projections.

Lemma (2.1.12)[99]: If p is a projection in $\mathcal{B}(H)$ with separable range, then $\Phi_*(p)$ is a projection with separable range and $\Phi(\dot{p}C(H)\dot{p}) = \Phi(\dot{p})C(H)\Phi(\dot{p})$.

Proof: Since a nonzero projection in $C(H)$ generates the minimal nontrivial ideal of $C(H)$ if and only if it is of the form $\dot{q}$ for some q with a separable range, the first claim follows. For the second part note that $A = \dot{p}C(H)\dot{p}$ is a hereditary subalgebra (i.e., if $0 \leq a \leq b$ for $a \in C(H)$ and $b \in A$, then $a \in A$) and therefore Φ maps it to a hereditary subalgebra.

A straightfoward recursive construction produces an increasing family of projections with separable range $p_{\alpha,}\ \alpha < \omega_1$ in $\mathcal{B}(H)$ such that

(i) $\bigvee_{\alpha<w_1} \mathrm{p}_\alpha = 1$ and for a limit δ we have $p_\delta = \bigvee_{\alpha<\delta}\mathrm{p}_\alpha$,

(ii) p_{Q} and each $p_{\alpha+1} — p_\alpha$ are noncompact,

(iii) for some projection r_α such that $\dot{r}_\alpha = \Phi(\dot{p}_\alpha)$ we have $p_\alpha \leq r_{\alpha+1}$ and $r_\alpha \leq p_{\alpha+1}$.

For convenience we write $\mathrm{p}_{-1} = 0$. For each α fix a basis of the range $\mathrm{p}_{\alpha+1} — \mathrm{p}_\alpha$ and enumerate it as e_β, for $\alpha\,.\,\omega \leq \beta < (\alpha + 1).\,\omega$. We therefore have a basis $(e_a)_{a<\omega_1}$ for H such that

(iv) p_α is the closed linear span of $\{e_\beta : \beta < \alpha\,.\,\omega\}$.

For every $\alpha < \omega_1$ Lemma (2.1.12) implies that the restriction of Φ to $\dot{p}_\alpha C(H)\dot{p}_\alpha$ is an isomorphism between Calkin algebras associated with separable Hilbert spaces, $p_\alpha[H]$ and $r_\alpha[H]$. Therefore by Theorem (2.1.1) we can fix a partial isometry v_α such that

(v) $v_\alpha v_\alpha^* \leq r_\alpha, v_\alpha^* v_\alpha \leq p_\alpha$, and Ad v_α is a representation of Φ on $\dot{p}_\alpha C(E)\dot{p}_\alpha$.

For each $\alpha > 1$ by Lemma (2.1.9) we can find $z_\alpha \in \mathbb{T}$ such that $v_0 - z_\alpha v_\alpha p_0$ is compact.

Replace v_α with $z_\alpha v_\alpha$ and note that Ad v_α still satisfies (vi). Let us prove that in addition (with $a =^{\mathcal{K}} b$ standing for '$a - b$ is compact')

(vi) $v_\alpha =^{\mathcal{K}} v_\beta p_\alpha$ whenever $\alpha < \beta$.

By Lemma (2.1.9), there is $z \in \mathbb{T}$ such that $v_a — z v_\beta p_a$ is compact. Since p_0 is non-compact and since $v_a p_0 =^{\mathcal{K}} v_0 =^{\mathcal{K}} v_\beta p_0$, we must have $z = 1$.

For $a \in \mathcal{B}(H)$ define the support of a as

$$supp(a) = \{\alpha < \omega_1 : ||ae_\alpha|| > 0\ or\ ||a^* e_\alpha|| > 0\}.$$

All compact operators are countably supported and the set of finitely supported operators is a dense subset of $\mathcal{K}(H)$. An easy analogue of the Δ-system lemma (e.g., [106]) is worth stating explicitly (here $H = \ell^2(\aleph_1)$ and p_α are as in (4)).

Lemma (2.1.13)[99]: Assume $a_\alpha, \alpha < \omega_1$, belong to $\mathcal{K}(H)$. Then for every $\varepsilon > 0$ there is a stationary $X \subseteq \omega_1, a$ finitely supported projection $r,$ and an operator a such α that $rar = a$ and

(a) $\|p_\alpha(ra_\alpha r — a_\alpha)p_\alpha\| < \varepsilon\ for\ all\ \alpha \in X,$

(b) $\|p_\alpha(a — a_\alpha)p_\alpha\| < \varepsilon\ for\ all\ \alpha \in X,, and$

(c) $||p_\alpha a_\alpha p_\alpha — p_\beta a_\beta p_\beta|| < 2\varepsilon$ for all $a < \beta$ in X.

Proof: For a_α find a finitely supported p_α with complex rational coefficients with support in p_α such that $\|p_\alpha(a_\alpha — b_\alpha)p_\alpha\| < \varepsilon/2$. By the Pressing Down Lemma ([106]) we can find a stationary set X_0 such that all b_α with $\alpha \in X_0$ have the same support, S. Let r be the projection to span$\{e_i : i \in S\}$. By a counting argument we can refine X_0 further and find a. The third inequality is an immediate consequence of the second.

For $\alpha < \omega_1$ let (with r_α and p_α as in (3))

$$X_\alpha = \{r_{\alpha+1}\omega p_\alpha : \omega \in \mathcal{B}(H), \omega =^{\mathcal{K}} v_\alpha\}.$$

Note the 'extra room' provided by defining X_α in this way instead of the apparently more natural $\{r_\alpha \omega p_\alpha : \omega \in \mathcal{B}(H), \omega =^{\mathcal{K}} v_\alpha\}$. Let us prove a few properties of X_α.

(vii) X_α is a norm-separable complete metric space.

(viii) If $\alpha < \beta$ then the map $\pi_{\beta\alpha}: X_\beta \to X_\alpha$ defined by

$$\pi_{\beta\alpha}(\omega) = r_{\alpha+1}\omega p_\alpha$$

is a surjection and a contraction.

Only the latter property requires a proof. It is clear that the range of $\pi_{\beta\alpha}$ is included in X_α and that the map is contraction. For $u \in X_\alpha$ let $\omega = v_\beta + u - r_{\alpha+1}v_\beta p_\alpha$. Then $\omega - v_\beta$ is compact since $u \in X_\alpha$ and clearly $r_{\alpha+1}\, \omega p_\alpha = r_{\alpha+1} u p_{\ \alpha} = u$.

Consider the Polish ω_1-tree T with levels X_α and connecting maps $\pi_\alpha \beta$.

Lemma (2.1.14)[99]: The following are equivalent.

(ix) Φ is inner.

(x)There is $a\ v \in \mathcal{B}$(H) such that $\dot{v}$ is a unitary in $C(H)$ and for all $\alpha < \omega_1$ we have

$$r_{\alpha+1} v p_\alpha \in X_\alpha.$$

(xi) T has a branch.

Proof: Clearly (10) and (11) are equivalent, hence it suffices to prove (9) implies (10) and that (10) implies (11). Assume Φ is inner and v implements it. Then by Lemma (2.1.9) for every $\alpha < \omega_1$ there is $z_\alpha \in \mathbb{T}$ such that $z_\alpha v p_\alpha - v_\alpha$ is compact. Since $v_\alpha\, p_0 - v_0$ is compact for each α and p_0 is noncompact, we have $z_\alpha = p_0$ for all α. Therefore $z_0 v$ defines a branch of T.

Now assume (11) and fix a v that defines a branch of T. Then the automorphism of $C(H)$ with representation $Ad\ v$ agrees with Φ on the ideal of all operators with separable range. By Lemma (2.1.10), this automorphism agrees with Φ on all of $C(H)$, hence (9) follows.

A minor modification of the proof that (10) implies (11) above gives an another equivalent reformulation of Φ being inner. Although we shall not need it, it deserves mention:

(xii) Every subtree of T h as a branch.

We proceed with the analysis of T and the corresponding 'local trees' $T[a]$.

For $b \in \mathcal{B}(H)$ and $\alpha < \omega_1$ let

$$Z[b]_\alpha = \{p_\alpha \omega b \omega^* p_\alpha : \omega \in X_{\alpha+1}\}.$$

Then for every $c \in Z[b]_\alpha$ we have $p_\alpha \Phi_*(b) p_\alpha =^{\mathcal{K}} c$ because

$$p_\alpha \omega b \omega^* p_\alpha =^{\mathcal{K}} p_\alpha \upsilon_{\alpha+1} p_{\alpha+1} b p_{\alpha+1} \upsilon^*_{\alpha+1} p_\alpha =^{\mathcal{K}} p_\alpha \Phi_*(p_{\alpha+1} b p_{\alpha+1}) p_\alpha$$
$$=^{\mathcal{K}} p_\alpha r_{\alpha+1} \Phi_*(b) r_{\alpha+1}\, p_\alpha =^{\mathcal{K}} p_\alpha \Phi_*(b) p_\alpha.$$

Also, for $\alpha < \beta$ the map $\overline{\omega}^b_{\beta\alpha}$ (denoted $\overline{\omega}\beta\alpha$ when b is clear from the context) from $Z[b]_\beta\ to\ Z[b]_\alpha$ defined by

$$\overline{\omega}\beta\alpha c) = p_\alpha C p_\alpha$$

is clearly a contractive surjection.

For $a \in \mathcal{B}(H)\ let\ T[a]$ denote the Polish ω_1-tree with levels $Z[a]_\alpha$ and commuting projections $\overline{\omega}\beta\alpha$. By 'subtree' we always mean a downwards closed subtree of height ω_1.

Lemma(2.1.15)[99]: For every $a \in \mathcal{B}$(H) every subtree S of $T[a]$ has a branch.
Proof: Let $b = \Phi_*(a)$. For every $\alpha < \omega_1$ fix $w_\alpha \in X_{\alpha+1}$ such that

$$b_\alpha = p_\alpha \omega_\alpha a \omega_\alpha^* p_\alpha$$

belongs to $S \cap Z[a]_\alpha$. Let $u_\alpha = p_\alpha \omega_\alpha$.

Fix $\varepsilon > 0$. Recall that the fixed basis e_a, for $a < \omega_1$, of H spans all p_α $(see$ (4)). Apply 'Δ-system' Lemma(2.1.13)to operators $p_\alpha(b - b_\alpha)p_\alpha$ to find uncountable $J \subseteq \omega_1$ and finitely supported c and $c_\alpha, \alpha \in J$, with disjoint supports, so that

$$\|(b - b_\alpha) - (c + c_\alpha)\| < \varepsilon \; and \; \|p_\alpha(b - b_\alpha)p_a - c\| < \varepsilon.$$

By going to a further subset of J we may assume that for $\alpha < \beta$ in J the support of c_α is included in $\beta . \omega$ (or more naturally stated, that $p_\beta c_\alpha p_\beta = c_\alpha$). For each $\alpha \in J$ let α^+ be the minimal element of J above α and let $b'_\alpha = p_\alpha(b_{\alpha+})p_\alpha$. For α in J we have $||b'_\alpha - (p_\alpha b p_\alpha - c)|| < \varepsilon$ and therefore $||b'_\alpha - p_\alpha b'_\beta p_\alpha|| < 2\varepsilon$ for $\alpha < \beta$ in J. Hence b'_α, for $\alpha \in J$, defines a 2ε -branch in $T[a]$. Since S has a 2ε -branch for an arbitrarily small ε it has a branch by Lemma (2.1.3).

We apply Martin's Axiom. First, we add a generic operator ${\rm T}$ to $\mathcal{B}(H)$ by a poset with finite conditions which forces that $T[{\rm T}]$ has a branch. Second, we use the properties of ${\rm T}$ to argue that T has a branch.

For a Hilbert space K with a fixed basis $e_j, j \in J$, let $\mathbb{p}(K)$ be the forcing defined as follows. A condition in $\mathbb{p}(K)$ is a pair (F, M) where F is a finite subset of J and M is an $F\ x\ F$ matrix with entries in the complex rationals, $\mathbb{Q} + i\mathbb{Q}$, such that the operator norm of M satisfies $\|M\| < 1$. We order $\mathbb{p}(K)$by extension, setting $(F', M') \leq (F, M)$ if $F' \supseteq F$ and $M' \restriction F\ x\ F \equiv M$.

Lemma (2.1.16)[99]: Poset $\mathbb{p}(K)$ is ccc if and only if K is separable.

Proof: if K is separable then $\mathbb{p}(K)$ is countable, so we only need to show the other direction. This direction will not be used in our proof, but we nevertheless include it since it shows why Lemma (2.1.17) below does not use $\mathbb{P}(H)$.

We may assume $0 \in J$. For each $j \in J \setminus \{0\}$ define a condition $a_j = (F^j, M^j)$ by $F^j = \{0, j\}$ and the $(0, j)$ entry of M^j is equal to $1/\sqrt{2}$, while the other three entries are 0. Then the norm of any matrix including M_j and M_k is at least 1, hence a_j, for $j \in J$, is an uncountable antichain.

By (2) in § (2.1.12) the projection

$$S_\alpha = P_{\alpha+1} - P_\alpha$$

has an infinite-dimensional and separable range. Let

$$\mathcal{D} = \{a \in \mathcal{B}(H) : a = \sum_{\alpha < w_1} s_\alpha a s_\alpha\}$$

where the sum is taken in the strong operator topology. This subalgebra of $\mathcal{B}(H)$ is an analogue of algebras $\mathcal{D}[\vec{E}]$ that played a prominent part in the proof of Theorem (2.1.1) in

[104]. Although much of the theory of $\mathcal{D}[\vec{E}]$ has analogues in the nonseparable case, we shall not develop this theory since the role of $\mathcal{D}$ in the proof of Theorem (2.1.11) is different.

For each $\alpha < \omega_1$ let $H_\alpha = s_\alpha H$, with the basis $\{e_\xi : \alpha . \omega \leq \xi < (\alpha + 1).\omega\}$ and let $\mathbb{P}_\alpha$ be $\mathbb{P}(H_\alpha)$. The finitely supported product $\mathbb{P}$ of $\mathbb{P}_\alpha$, for $\alpha < \omega_1$is ccc. Actually, being a finitely supported product of countable posets, it is forcing-equivalent to the poset for adding $\aleph_1$ Cohen reals.

If $\dot{G} \subseteq \mathbb{P}$ is a generic filter, then it defines a sesquilinear form whose norm is, by genericity, equal to 1. This in turn defines an operator on H in the unit ball of $\mathcal{B}(H)$ ([108]) This operator belongs to the von Neumann algebra $\mathcal{D}$ and we let т denote its $\mathbb{P}$-name.

Lemma(2.1.17)[99]: Poset $\mathbb{P}$ forces that every subtree of T[т] has a branch.

Proof: If not, then by Lemma (2.1.15) we fix a condition $p \in \mathbb{P}$ deciding $\varepsilon > 0$ such that some subtree T'[т] of T [т] has no ε-branch and consider $\mathbb{P} * \dot{\mathbb{S}}$ (below p) where $\dot{\mathbb{S}}$ is a ccc poset for $\varepsilon/2$-specializing T'[т]. By applying MA we can find $a \in \mathcal{B}(H)$ and an $\varepsilon/2$-special subtree of $T[a]$. By Lemma (2.1.4) this subtree has no branches, and this contradicts Lemma (2.1.15).

Fix $\varepsilon > 0$. By Lemma (2.1.17), if S is a subtree of T then for $\alpha < \omega_1$ we can fix ω_α and a condition $\mathrm{a}_{\boldsymbol{\alpha}}$ in $\mathbb{p}$ that forces $\mathrm{Ad}(\mathrm{p}_\alpha \omega_\alpha)$ т belongs to a cofinal ε-branch of T[т]. Here $w_\alpha \in S \cap X_{\alpha+1}$ and ω_α is in the ground model. Identify $\mathrm{a}_{\boldsymbol{\alpha}}$ with a finitely supported operator in $\mathcal{B}(H)$ and note that it belongs to the algebra $\mathcal{D}$ as defined. Apply Lemma (2.1.13) to $\{\mathrm{Ad}(\mathrm{p}_\alpha \mathrm{w}_\alpha)\mathrm{a}_{\boldsymbol{\alpha}}\}$ to find a finitely supported b such that

(xiii) $||b - \mathrm{Ad}(\mathrm{p}_\alpha \omega_\alpha)\mathrm{a}_{\boldsymbol{\alpha}}|| < \varepsilon$

for all α in a stationary set J_0. Since the coefficients of $\mathrm{a}_{\boldsymbol{\alpha}}$ are complex rationals, by the Δ-system lemma and a counting argument there are a stationary set $\mathrm{J}_1 \subseteq \mathrm{J}_0$, a finitely-supported projection q, and a such that

(xiv) $\mathrm{qaq} = \mathrm{a}$ and $p_{\boldsymbol{\alpha}} \mathrm{a}_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} = \mathrm{a}$

for all $\alpha \in J_1$. Note that $\mathrm{a}_{\boldsymbol{\alpha}} = \mathrm{a} + (\mathrm{I} - p_{\boldsymbol{\alpha}}) a_{\boldsymbol{\alpha}} (I - p_{\boldsymbol{\alpha}})$ for all $\alpha \in J_1$. Find $\bar{\alpha}$ such that $P_{\bar{\alpha}} q = q$. Applying Lemma (2.1.10) to $(\omega_\beta - \nu_{\bar{\alpha}})\mathrm{p}_{\bar{\alpha}}$ find a stationary $J \subseteq J_1$ such that

(xv) $||(\omega_\beta - \omega_\gamma)\mathrm{p}_{\bar{\alpha}}|| < \varepsilon$

for all $\beta < \gamma$ in J. Let $q_{\boldsymbol{\alpha}}$ denote the support of $\mathrm{a}_{\boldsymbol{\alpha}}$. For $\beta \in J$ let $u_\beta = \omega_\beta p_\beta$. Then for $\alpha + 1 \leq \beta$ we have $p_\alpha u_\beta =^{\mathcal{K}} p_{\boldsymbol{\alpha}} \omega_\beta$.

Preparations for the proof of Lemma(2.1.22)take up the remainder, with the main points being Claim(2.1.20) and Lemma(2.1.21).

Claim(2.1.18)[99]: If $a \in \mathcal{D}, \alpha < \beta$ are in $J, q_\alpha a q_{\ \alpha} = \mathrm{a}_\alpha$, and $q_\beta a q_\beta = \mathrm{a}_{\boldsymbol{\beta}}$, then

$$\|Ad(p_\alpha \omega_\alpha)a - Ad(p_\alpha \omega_\beta)a\| \leq \varepsilon.$$

Proof: Otherwise, there is $\delta > 0$ and a finitely supported projection $s \geq q_\alpha \vee q_\beta$ such that for every $c \in \mathcal{D}$ satisfying $scs = sas$ we have $\left\|Ad(p_\alpha \omega_\alpha)c - Ad(p_\alpha \omega_\beta)c\right\| > \varepsilon + \delta$.

Making a small change to coefficients of sas one obtains a condition in $\mathbb{P}$ forcing that $\|Ad(p_\alpha \omega_\alpha)T - Ad(p_a \omega_\beta)$т$\| > \varepsilon,$ a contradiction

Claim(2.1.19)[99]: Assume a and b are in $\mathcal{D}, qaq = qbq = 0, p_\alpha a p_{\alpha+\omega} = p_\alpha b p_{\alpha+w}$, and $\alpha + \omega < \beta \;\; for\, \beta \in J$. Then

$$\|Ad(p_\alpha \omega_\beta)(a + \mathrm{a}_{\boldsymbol{\beta}}) - Ad(p_\alpha \omega_\beta)(b + \mathrm{a}_{\boldsymbol{\beta}})\| \le 2\varepsilon.$$

Proof: Assume otherwise and let

$$\delta = \|\mathrm{Ad}(p_\alpha \omega_\beta)(a + \mathrm{a}_{\boldsymbol{\beta}}) - Ad(p_\alpha \omega_\beta)(b + \mathrm{a}_{\boldsymbol{\beta}})\| - 2\varepsilon.$$

For $n < \omega$ write $s_n = p_{\alpha_{+\omega}} - p_{\alpha+n}$. By continuity fix $n < \omega$ such that for all $c \in \mathrm{s_n}\mathcal{D}$ ($= s_n \mathcal{D} s_n$ *since* s_n in the commutant of $\mathcal{D}$) with $\|c\| \le 1$ we have

$$\|Ad(p_\alpha \omega_\beta)(a + \mathrm{a}_{\boldsymbol{\beta}}) - Ad(p_\alpha \omega_\beta)(1 - s_n)(a + \mathrm{a}_{\boldsymbol{\beta}}) + c)\| < \delta/2$$

and

$$\| Ad(p_\alpha \omega_\beta)(a + \mathrm{a}_{\boldsymbol{\beta}}) - Ad(p_\alpha \omega_\beta)(1 - s_n)(b + \mathrm{a}_{\boldsymbol{\beta}}) + c)\| < \delta/2.$$

Let $c = \mathrm{a}_{\boldsymbol{\alpha+n}} - \mathrm{a}$. Then Claim (2.1.18). applied to $(1 - s_n)(a + \mathrm{a}_{\boldsymbol{\beta}}) + c$ and to $(1 - s_n)(b + \mathrm{a}_{\boldsymbol{\beta}}) + c$ implies

$$\left\|Ad(p_\alpha \omega_\beta)\left((1 - s_n)(a + \mathrm{a}_{\boldsymbol{\beta}}) + c\right) - Ad(p_\alpha \omega_{\alpha+n})\left((1 - s_n)(a + \mathrm{a}_{\boldsymbol{\beta}}) + c\right)\right\| \le \varepsilon$$
$$\left\|Ad(p_\alpha \omega_\beta)\left((1 - s_n)(a + \mathrm{a}_{\boldsymbol{\beta}}) + c\right) - Ad(p_\alpha \omega_{\alpha+n})\left((1 - s_n)(b + \mathrm{a}_{\boldsymbol{\beta}}) + c\right)\right\| \le \varepsilon$$

leading to $2\varepsilon + \delta < 2\varepsilon + \delta$.

Claim (2.1.20)[99]: For $\alpha + \omega < \beta < \gamma$ such that β and γ are in J we have

$$\Delta = \|Ad(p_\alpha u_\beta)a - Ad(p_\alpha u_\gamma)a\| \le 5\varepsilon$$

for all $a \in \mathcal{D}$ with $\|a\| \le 1$ and $(1 - p_\beta)a = 0$.

Proof: Fix $a \in \mathcal{D}$ with $\|a\| \le 1$. We have that $\boldsymbol{c} = \mathrm{a}_{\boldsymbol{\beta}} + (1 - p_\gamma)\, \mathrm{a}_\gamma$ is a condition in $\mathbb{P}$ with support $q' = q_\beta \, V \, q_\gamma$ extending both $\mathrm{a}_{\boldsymbol{\beta}}$ and $a_{\boldsymbol{\gamma}}$. Let

$$a' = a - q' a q' + \boldsymbol{c}.$$

With $\bar{\alpha}$ as in (15) we have $p_{\bar{\alpha}} a = a p_{\bar{\alpha}}$ since $a \in \mathcal{D}$. Therefore

$$\begin{aligned} Ad(p_\alpha u_\beta)a - Ad(p_\alpha u_\beta)a' &= Ad(p_\alpha u_\beta p_{\bar{\alpha}})(a - a') + Ad(p_\alpha u_\beta (p_\beta - p_{\bar{\alpha}}))(a - a') \\ &= Ad(p_\alpha u_\beta p_{\bar{\alpha}})(a - a'). \end{aligned}$$

By this and an analogous computation for γ we have

$$\begin{aligned} Ad(p_\alpha u_\beta)a - Ad(p_\alpha u_\gamma)a = {} & Ad(p_\alpha u_\beta p_{\bar{\alpha}})(a - a') - Ad(p_\alpha u_\gamma p_{\bar{\alpha}})(a - a') \\ & + Ad(p_\alpha u_\beta)a' - Ad(p_\alpha u_\gamma)a' \end{aligned}$$

Using (15) and $p_\beta\, \mathrm{a}_{\boldsymbol{\beta}} = p_\gamma\, \mathrm{a}_{\boldsymbol{\gamma}} = \mathrm{a}$ we conclude that each of the first two summands has norm $\le \varepsilon$, hence Δ is within 2ε of $\|Ad(p_\alpha u_\beta)a' - Ad(p_\alpha u_\gamma)a'\|$. *Since* $a' \in \mathcal{D}$ we have $(1 - p_\beta)a' = (1 - p_\beta)\mathrm{a}_{\boldsymbol{\beta}}$ and the following.

$$Ad(p_\alpha u_\beta)a' = Ad(p_\alpha \omega_\beta)a' - Ad(\omega_\beta (1 - p_\beta))\mathrm{a}_{\boldsymbol{\beta}}.$$

By this and an analogous computation for γ we have

$$\begin{aligned} Ad(p_\alpha u_\beta)a' - Ad(p_\alpha u_\gamma)a' = {} & Ad(p_\alpha \omega_\beta)a' - Ad(p_\alpha \omega_\gamma)a' \\ & + Ad\left(\omega_\beta (1 - p_\beta)\right) a_{\boldsymbol{\beta}} - Ad(\omega_\gamma (1 - p_\gamma))\mathrm{a}_{\boldsymbol{\gamma}}. \end{aligned}$$

By Claim (2.1.18) the first difference has norm $\le \varepsilon$ and by (13) the second difference has norm $\le 2\varepsilon$. The conclusion follows.

We are now within definitions and computations from completing the proof. In order to complement Claim(2.1.20) in the proof of Lemma (2.1.22), we digress a little bit. For $\alpha < \omega_1$ define the following metrics on $X_{\alpha+1}$ (only d_4 and d_2 will be needed in our proof).

$$d_{1,\alpha}(u,\omega) = \|u - \omega\|$$

$$d_{2,\alpha}(u,\omega) = \sup_{a\in\mathcal{D},\|a\|=1} \| Ad\, ua - Ad\, \omega a\|$$

$$d_{2,\alpha}(u,\omega) = \sup_{a\in\mathcal{B}(H),\|a\|=1} \| Ad\, ua - Ad\, \omega a\|$$

$$d_{4,\alpha}(u,\omega) \|p_\alpha (u - \omega)\|$$

We shall drop the subscript a whenever it is clear from the context.

Lemma (2.1.21)[99]: For all α, on $X_{\alpha+1}$ we have $d_4 \le \mathrm{d}_2 \le \mathrm{d}_3 \le 2\mathrm{d}_1$.

Proof: The inequality $d_2 \le d_3$ is trivial, and $d_3 \le 2d_1$ follows from the following computation.

$$\|Ad\, ua - Ad\, wa\| \le \|uau^* - uaw^*\| + \|uaw^* - waw^*\|$$
$$\le \|ua\|.\|u^* - w^*\| + \|u - w\| . \|ua\|$$

It remains to prove $d_4 \le d_2$.

Let $v,w \in X_{\alpha+1}$ be given, and put $d = \|p_\alpha(v - w)\|$. Fix $\delta > 0$ and a unit vector ξ such that $\|(v^* - w^*)p_\alpha\xi\| > d - \delta$. Clearly we may assume $p_\alpha\xi = \xi$. Let ζ be a unit vector colinear with $v^*\xi - w^*\xi$ and let ι be a unit vector orthogonal to ζ such that $v^*\xi$ and $w^*\xi$ belong to the linear span of ζ and ι. Fix scalars $x, y, x'y'$ such that

$$v^*\xi = x\zeta + y\iota$$
$$w^*\xi = x'\xi + y'\iota$$

Since $v^*\xi - w^*\xi$ is colinear with ζ, we have $y = y'$. Therefore $\|v^*\xi - w^*\xi\| = |x - x'|$.

Find representations $\zeta = \sum_{\gamma<\alpha} x_\gamma\zeta_\gamma$ and $\iota = \sum_{\gamma<\alpha} y_\gamma\iota_\gamma$ so that ζ_γ and ι_γ belong to the range of $s_\gamma = p_{\gamma+1} - p_\gamma$ for all γ. Since the range of s_γ is infinite-dimensional and since $v - w$ is compact, we can find a unit vector v_γ in this range orthogonal to both ζ_γ and ι_γ and such that $\|vv_\gamma\| = 1$ but $\|vv_\gamma - wv_\gamma\| < \delta/d$. Let

$$v = \sum_{\gamma<\alpha} x_\gamma v_\gamma$$

Then ζ, ι, and v are mutually orthogonal unit vectors and the rank two operator $a \in \mathcal{B}(H)$ defined by $a(v) = \zeta$ and $a(\zeta) = v$ has norm equal to one. Moreover, $a \in \mathcal{D}$, since for each γ the operator $as_\gamma = s_\gamma a$ is just the rank-two operator which transposes the orthogonal unit vectors v_γ and ζ_γ. Note that $((Ad\, v)a)\xi = vav^*\xi = va(x\zeta + y\iota) = xwv$ and $((Adw)a)\xi = waw^*\xi = wa(x'\zeta + y\iota) = x'wv$. Hence,

$$\|((Adv)a - (Adw)a)\xi\| = \|(x - x')wv\| = |x - x'| > d - \delta.$$

Since $\delta > 0$ was arbitrary, we conclude that $d_2(v,w) \ge d$.

Lemma(2.1.22)[99]: The set $\{r_{\alpha+2}u_\beta p_{\alpha+1} : \alpha + \omega < \beta, \beta \in J\}$ *is a* 5ε-branch of T.

Proof. In order to show $\{r_{\alpha+2}u_\beta p_{\alpha+1} : \alpha + \omega < \beta, \beta \in J\}$ is a 5ε-branch, it suffices to show that $\|p_{\alpha+3}(u_\beta - u_\beta)p_{\alpha+2}\| \le 5\varepsilon$ whenever $\alpha + \omega < \beta < \gamma$ for β, γ in J. But the inequality $d_{4,\alpha+1} \le d_{2,\alpha+1}$ from Lemma (2.1.21) implies

$$\|P_{\alpha+3}(u_\beta - u_\gamma)P_{\alpha+2}\| \le \sup_{a \in \mathcal{D}} \|Ad(p_{\alpha+3}u_\beta p_{\alpha+2})a - Ad(p_{\alpha+3}u_\gamma p_{\alpha+2})\, a\|$$

and the right hand side is $\le 5\varepsilon$ by Claim (2.1.20)

Since ε was arbitrary, Lemma (2.1.22) and Lemma (2.1.3) imply that T has a cofinal branch. By Lemma (2.1.14), Φ is inner.

Theorem (2.1.23)[99]: PFA implies all automorphisms of every Calkin algebra are inner.

The only use of TA in the present is implicit via the following result from [104].

Proof. The proof of Theorem (2.1.23) is reasonably similar to the proof of the analogous result from [112]. All we need is the analysis of coherent families of Polish spaces and a fragment of PFA. Fix $k > \aleph_2$, write $H = \ell^2(\kappa)$ and let Φ be an automorphism of the Calkin algebra $C(H)$. Fix a basis $\{e_\alpha : \alpha < k\}$ of H and denote the projection to $\overline{span\{e_\alpha : \alpha \in \lambda\}}$ byp_λ

Recall that $\mathcal{P}_{w_1}(\kappa)$ denotes the family of all countable subsets of k. This set is σ −directed under the inclusion and it is a lower semilattice. For every countable subset $\lambda \subseteq k$ fix projection $r_\lambda$with separable range such that $\Phi(\dot{p}_\lambda) = \dot{r}_\lambda$. For $\lambda < \lambda'$ in Λ we have $\dot{r}_\lambda \le \dot{r}_{\lambda'}$ but not necessarily $r_\lambda \le \dot{r}_{\lambda'}$. By [104] we can fix a partial isometry v_λ such that Ad v_λ implements the restriction of Φ to $\dot{p}_\lambda C(H)\dot{p}_\lambda$. For $\lambda \in \mathcal{P}_{w_1}(k)$ let

$$X_\lambda = \{r_\lambda w p_\lambda : w \in \mathcal{B}(H), w =^{\mathcal{K}} v_\lambda\}.$$

Let us prove a few properties of X_λ.

(xvi) X_λ is a norm-separable complete metric space.

(xvii) If $\lambda \subseteq \lambda'$then the map $\pi_{\lambda'\lambda} : X_{\lambda'} \to \lambda_\lambda X_v$ defined by

$$\pi_{\lambda'\lambda}(w) = r_\lambda w p_\lambda$$

is a contraction.

The proof is analogous to the proof of (viii). Consider the coherent family of Polish spaces

$$\mathbb{F} = (X_\lambda, \pi_{\lambda'\lambda}, \pi_{\lambda'\lambda}, \text{for } \lambda \in \mathcal{P}_{w_1}(k)).$$

The omitted proof of the following uses Lemma (2.1.10) and is analogous to the proof of Lemma (2.1.14).

Lemma (2.1.24)[99]: The following are equivalent.

(xviii) Φ is inner.

(xix) There is $v \in \mathcal{B}(H)$ such that $\dot{v}$ is a unitary in $C(H)$ and for all $\lambda \in \mathcal{P}_{w_1}\kappa(k)$ we have $r_\lambda v p_\lambda \in X_\lambda$.

(xx) The coherent family of Polish spaces $\mathbb{F}$ is trivial.

If Φ is not inner, then by Lemma (2.1.24) and Lemma (2.1.6) there is an $\boldsymbol{\varepsilon} > 0$ and a cofinal subfamiy $\mathbb{F}'$ of $\mathbb{F}$ with no $\boldsymbol{\varepsilon}-$ branches. By PFA and Lemma (2.1.7), there is a strictly increasing map $f: w_1 — \mathbb{F}$ such that the Polish w_1-tree $(X_{f(\alpha)}, d_{f(\alpha)}, \pi_{f(\beta)f(\alpha)}, \alpha \leq \beta < w_1)$ is $\varepsilon/6$ $-$special. Then $Z = \bigcup f[w_1]$ is an $\aleph_1$-sized subset of k. Let $C(Z)$ denote the Calkin algebra associated with $\mathcal{B}(\ell^2(Z))$. By modifying the proof of Lemma (2.1.7) and meeting some additional dense sets, we can assure that the restriction Φ_z of Φ to $C(Z)$ is an automorphism of $C(Z)$.

Theorem (2.1.11) implies Φ_z is inner and Lemma (2.1.14) implies Φ_z is outer. This contradiction concludes the proof of Theorem (2.1.23).

Corollary(2.1.25)[370]: The set $\{r_{\alpha^2+2}u_{\beta^2}p_{\alpha^2+1} : \alpha^2 + \omega < \beta^2, \beta^2 \in J\}$ *is a* 5ε-branch of T.

Proof. In order to show $\{r_{\alpha^2+2}u_{\beta^2}p_{\alpha^2+1} : \alpha^2 + \omega < \beta^2, \beta^2 \in J\}$ is a 5ε-branch, it suffices to show that $\|p_{\alpha^2+3}(u_{\beta^2} — u_{\beta^2})p_{\alpha^2+2}\| \leq 5\varepsilon$ whenever $\alpha^2 + \omega < \beta^2 < \gamma^2$ for β^2, γ^2 in J. But the inequality $d_{4,\alpha^2+1} \leq d_{2,\alpha^2+1}$ from Lemma (2.1.21) implies

$$\|P_{\alpha^2+3}(u_{\beta^2} - u_{\gamma^2})P_{\alpha^2+2}\| \leq \sup_{a\in\mathcal{D}}\|Ad(p_{\alpha^2+3}u_{\beta^2}p_{\alpha^2+2})a - Ad(p_{\alpha^2+3}u_{\gamma^2}p_{\alpha^2+2})\, a\|$$

and the right hand side is $\leq 5\varepsilon$ by Claim (2.1.20).

Section (2.2): Corona Algebras

We shall investigate the degree of countable saturation of coronas (see Definition (2.2.2) and paragraph following it). This property is shared by ultra products associated with no principal ultra filers on $\mathbb{N}$ in its full form. The following summarizes the results. All ultra filters are no principal ultra filters on $\mathbb{N}$.

Theorem (2.2.1)[114]: Assume a C^*-algebra M is in one of the following forms:

(i) the corona of a σ-unital C^*-algebra,
(ii) an ultraproduct of a sequence of C^*-algebras,
(iii) an ultrapower of a C^*-algebra,
(iv) $\prod_n A_n/\bigotimes_n A_n$, for unital C^*-algebras A_n,
(v) the relative commutant of a separable subalgebra of an algebra that is in one of the forms (i)-(iv).

Then M satisfies each of the following (see below for definitions):

(vi) It is SAW*
(vii) It has *AA-CRISP* (asymptotically abelian, countable Riesz separation property),
(viii) The conclusion of Kasparov's technical Theorem,
(ix) It is sub-a-Stonean in the sense of Kirchberg,
(x) Every derivation of a separable subalgebra of M is of the form δ_b for some b $\in$ M.

Proof. Each of these classes of C^*-algebras is countably degree-1 saturated (Definition (2.2.2). For (i) this is Theorem (1.2.4). For (ii) and (iii) this is a consequence of Los's Theorem (see e.g., [104]). Every algebra as in (iv) is the corona of $\bigoplus_n A_n$so this is a special case of (i). For (v) this is Lemma (2.2.9).

Property (vi) now follows by Proposition (2.2.12), (vii) follows by Proposition (2.2.12), (viii) follows by Proposition (2.2.13), (ix) follows by Proposition (2.2.16), and (x) follows by Proposition (2.2.17).

The assertion 'every approximately inner automorphism of a separable subal-gebra of M is implemented by a unitary in M′ ′ is true for algebras as in (ii), (iii) or the corresponding instance of (iv) Lemma (2.2.19). However this is not true in the case when M is the Calkin algebra (see Proposition (2.2.27)).

By [121] no SAW*-algebra can be written as a tensor product of two infinite-dimensional C*-algebras. By Theorem (2.2.1), this applies to every C*-algebra M satisfying any of (i)-(v).

We demonstrate that the degree of saturation of the Calkin algebra is rather mild.

For $F \subseteq \mathbb{R}$ and $\varepsilon > 0$ we write $F_\varepsilon = \{x \in \mathbb{R} : \operatorname{dist}(x, F) \le \varepsilon\}$. Given a $C*$-algebra A, a degree 1*-polynomial in variables x_j, for $j \in \mathbb{N}$, with coefficients in A is a linear combination of terms of the form ax_jb, ax_j^*b and a with a, b in A. We write $M_{\le 1}$ for the unit ball of a C^*-algebra M.

Definition (2.2.2)[114]: A metric structure M is countably degree-*1* saturated if for every countable family of degree-1* -polynomials $P_n(\bar{x})$ with coefficients in M and variables x_n, for $n \in \mathbb{N}$, and every family of compact sets $K_n \subseteq \mathbb{R}$, for $n \in \mathbb{N}$, the following are equivalent.

(i) There are $b_n \in M_{\le 1}$, for $n \in \mathbb{N}$, such that $P_n(\bar{b}) \in K_n$ for all n.
(ii) For every $m \in \mathbb{N}$ there are $b_n \in M_{\le 1}$,, for $n \in \mathbb{N}$, such that $P_n(\bar{b}) \in (K_n)_{1/m}$ for all $n \le m$.

More generally, if Φ is a class of $*$-polynomials, we say that M is countably Φ-saturated if for every countable family of $*$-polynomials $P_n(\bar{x})$ in Φ with coefficients in M and variables x_n, for $n \in \mathbb{N}$, and every family of compact sets $K_n \subseteq \mathbb{R}$, for $n \in \mathbb{N}$ the assertions (i) and (ii) above are equivalent.

If Φ is the class of all $*$-polynomials then instead of Φ -saturated we say count-ably quantifier-free saturated.

By compactness we obtain an equivalent definition if we require each K_n to be a singleton.

With the obvious definition of degree-n saturated' one might expect to have a proper hierarchy of levels of saturation.

Lemma (2.2.3)[114]: An algebra that is degree-2 saturated is necessarily quantifier-free saturated.

Proof. Assume C is degree-2 saturated and t is a consistent countable quanti-fier free type over C. By compactness and the Stone-Weierstrass approximation Theorem we may assume that t consists of formulas of the form $\|P(\bar{x})\| = r$ for a polynomial P. By adding a countable set of new variables $\{z_i\}$and formulas $\|xy - Z_i\| = 0$ for distinct variables x and y occurring in t, one can reduce the degree of all polynomials occurring in t. By repeating this procedure countably many times one obtains a new type t in countably many variables such that t' does not contain polynomials of degree higher than 2, it is consistent, and a realization of t' gives a realization of t.

In the following it is assumed that each P_n is a $*$polynomial with coefficients in M, and reference to the ambient algebra M is omitted whenever it is clear from the context. An expression of the form $P_n(\bar{x}) \in K_n$ is called a condition (over M). A set of conditions is a type (over M). If all conditions involve only polynomials in Φ then we say that the type is a Φ -type. If all coefficients of polynomials occurring in type t belong to a set $X \subseteq M$ then we say t is a type over X. A type satisfying (2) is approximately finitely satisfiable (in M), or more succinctly consistent with M, and a type satisfying (1) is realized (in M) by b. In the latter case we also say that M realizes this type. Thus M is countably Φ-saturated if and only if every consistent Φ-type over a countable subset of M is realized in M.

Recall that the multiplier algebra $M(A)$ of a C^*-algebra A is defined to be the idealizer of A in any nondegenerate representation of A (see e.g., [100]). The corona of A is the quotient $M(A)/A$.

Corollary (2.2.4)[114]: If A is α σ-unital C^*-algebra then $M_n(C(A))$ is countably degree-1 saturated for every $n \in \mathbb{N}$.

Proof. The universality property of the multiplier algebra easily implies that $M(M_n(A))$ and $M_n(M(A))$ are isomorphic, via the natural isomorphism that fixes A. Therefore $M_n(C(A))$ is isomorphic to $C(M_n(A))$ and we can apply Theorem (2.2.26).

The following will be proved as Theorem (2.2.23).

Theorem (2.2.5)[114]: Assume A is a σ-unital C^*-algebra such that for every separable subalgebra B of $M(A)$ there is a B-quasicentral approximate unit for A consisting of projections. Then its corona $C(A)$ is countably quantifier-free saturated.

We shall show that the Calkin algebra fails the conclusion of Theorem (2.2.5), and therefore that Theorem (2.2.42) essentially gives an optimal conclusion in its case.

Most of the applications require only types with a single variable, or so-called *1*-types. We shall occasionally use shortcuts such as $a = b$ for $\|a - b\| = 0$ or $a \leq b$ for $b - a$ being positive (the latter assuming both a and b are positive) in order to simplify the notation. We say that c ε-realizes type t if for all conditions $\|P(x)\|t \in K$ in t we have $\|P(c)\| \in (K)_\varepsilon$. Therefore a type is consistent if and only if each of its finite subsets is ε-realized for each $\varepsilon > 0$.

We start with a self-strengthening of the notion of approximate finite satisfiability, stated only for 1-types.

Lemma (2.2.6)[114]: If Φ includes all degree-1 *-polynomials and C is countably Φ-saturated then every countable Φ-type t that is approximately finitely satisfiable by self-adjoint (positive) elements is realized by a self-adjoint (positive) element.

Moreover, if t is approximately finitely satisfiable by self-adjoint elements whose spectrum is included in the interval $[r, s]$, then t is realized by a self-adjoint element whose spectrum is included in $[r, s]$.

Proof. If t is approximately finitely satisfiable by a self-adjoint element, then the type $t_{\mathbf{1}}$ obtained by adding $x = x^*$ to t is still approximately finitely satisfi-able and countable, and therefore realized. Any realization of $t_{\mathbf{1}}$is a self-adjoint realization of t**.**

Now assume t is approximately finitely satisfiable by positive elements. By compactness, there is $r \in K$ such that $t \cup \{\|x\| = r\}$ is approximately finitely satisfiable by a positive element. Let $t_{\mathbf{2}} = t \cup \{\|x\| = r, x = x^*, \|x - r\,.\,1\| \leq r\}$. A simple continuous functional calculus argument shows that for a self-adjoint b we have that $b \geq 0$ if and ony if $\|b - \| \ \|b\|\|.1\| \ \ \|b\|$. The proof is completed analogously to the case of a self-adjoint operator.

Now assume t is approximately finitely satisfiable by elements whose spectrum is included in $[r, s]$. Add conditions $\|x - x^*\| = 0$ and $\|x - (r + s)/2\| \leq (s - r)/2$ to t**.** The second condition is satisfied by a self-adjoint element iff its spectrum is included in the interval $[r, s]$. Therefore the new type is approximately finitely satisfiable and its realization is as required.

The assumption of Lemma (2.2.6) is necessarily stronger than the assumption of Lemma (2.2.6).

Lemma (2.2.7)[114]: If C is countably quantifier-free saturated then every countable quantifier-free type that is approximately finitely satisfiable by a unitary (projection) is realized by a unitary (projection, respectively).

Proof. This is just like the proof of Lemma (2.2.6), but adding conditions $xx^* = 1$ and $x^*x = 1$ in the unitary case and $x = x^*$ and $x^2 = x$ in the projection case.

In Proposition (2.2.27) and Proposition (2.2.28) we prove that there is a countable type over the Calkin algebra that is approximately finitely satisfiable by a unitary but not realized by a unitary. By Lemma(2.2.7), the Calkin algebra is not quantifier-free saturated.

Largeness of countably saturated C*-algebras If C is a finite-dimensional C*-algebra then its unit ball is compact, and this easily implies that C is count-ably saturated.

Proposition (2.2.8)[114]: If C is countably degree-1 saturated then it is either finite-dimensional or nonseparable. In the latter case, C even has no separable maximal abelian subalgebras.

Proof. Assume C is infinite-dimensional and let A be its masa. Then A is infinite-dimensional and there is a sequence of positive operators a_n, for n G $\mathbb{N}$, of norm 1 such that $\|a_m - a_n\| = 1$ (cf. [126] or [120]).

Assume A is separable, and fix a countable dense subset b_n, for $n \in \mathbb{N}$, of its unit ball. The type t consisting of all conditions of the form $\|x - b_n\| \geq 1/2$ and $xb_n = b_nx$, for $n \in N$, together with $\|x\| = 1$, is consistent. This is because each of its finite subsets is realized by a_m for a large enough m. Otherwise, there are n, i and j such that $\|b_n - a_i\| < 1/2$ and $\|b_n - a_i\| < 1/2$. By countable saturation some $c \in C$ realizes t**.** Then $c \in A' \setminus A$, contradicting the assumed maximality of A.

Lemma (2.2.9)[114]: Assume C is countably Φ-saturated and Φ includes all degree-1 polynomials. If A is a separable subalgebra of C then the relative commutant of A is countably Φ -saturated.

Moreover, if C is infinite-dimensional then $A' \cap C$ is nonseparable.

Proof. Let a_n, for $n \in \mathbb{N}$, enumerate a countable dense subset of the unit ball of A. The relative commutant type over A, t_{rc}, consists of all formulas of the form

(i) $\|a_n x - x a_n\| = 0$, for $n \in \mathbb{N}$.

If t is a finitely approximately finitely satisfiable Φ-type over $A' \cap C$ then $t \cup t_{rc}$ is an approximately finitely satisfiable Φ -type over C. Also, an element c of C realizes $t \cup t_{rc}$ if and only if $c \in A' \cap C$ and c realizes t**.** Since t was an arbitrary Φ-type, countable Φ-saturation of $A' \cap C$ follows.

Now assume C is infinite-dimensional. By enlarging A if necessary, we can assume it is infinite-dimensional. Expand t_{rc} by adding all formulas of the form

(ii) $\|a_n x - a_n\| \geq 1/2$.

We denote the resulting type by t**.** We shall prove that t is approximately finitely satisfiable. This follows from the proof of [120]. First, if A is a continuous trace, infinite-dimensional algebra then its center $Z(A)$ is infinite-dimensional. Therefore Z(A) includes a sequence of contractions f_n, for $n \in \mathbb{N}$, such that $\|f_m - f_n\| = 1$ if $m \neq n$ (this is a consequence of Gelfand-Naimark Theorem, see e.g., the proof of [120]), and therefore t is approximately finitely satisfiable by $f_m s$.

If A is not a continuous trace algebra, then by [1] it has a nontrivial central sequence. Elements of such a sequence witness that t is approximately finitely satistiable.

By countable saturation, t is realized in $C. A$ realization of t in C is at a distance $\geq 1/2$ from A, and therefore we have proved that $A' \cap C \nsubseteq A$.

Now assume A is a separable, not necessarily infinite-dimensional, subalgebra of C. Since C is infinite-dimensional, find infinite-dimensional A_0 such that $A \subseteq A_0 \subseteq C$. By using the above, build an increasing chain of separable subalgebras of C, A_γ, for $\gamma < \aleph_1$, such that $A'_\gamma \cap A_{\gamma+1}$ is nontrivial for all γ. This shows that $A' \cap C$ intersects $A_{\gamma+1} \setminus A_\gamma$ for all γ, and it is therefore nonseparable.

In the following there is a clear analogy with the theory of gaps in $\mathcal{P}(\mathbb{N})/F$in.

Definition (2.2.10)[114]: Two subalgebras A, B of an algebra C are orthogonal if $ab = 0$ for all $a \in A$ and $b \in B$. They are separated if there is a positive element $c \in C$ such that $cac = a$ for all $a \in A$ and $cb = 0$ for all $b \in B$.

A C^*-algebra C has AA-CRISP (asymptotically abelian, countable Riesz separation property) if the following holds: Assume a_n, b_n, for $n \in \mathbb{N}$, are positive elements of C such that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

for all n. Furthermore assume D is a separable subset of C such that for every $d \in D$ we have

$$\lim_n \|a_n, d\| = 0.$$

Then there exists a positive $c \in C$ such that $a_n \leq c \leq b_n$ for all n and $[c, d] = 0$ for all $d \in D$.

By Theorem (2.2.26) the following is a strengthening of the result that every corona of a σ-unital C^*-algebra has AA-CRISP ([127]).

Proposition (2.2.11)[114]: Every countably degree-1 saturated C*-algebra C has AA-CRISP.

Proof. By scaling, we may assume that $\|b_1\| = 1$. Fix a countable dense subset $\{d_n\}$ of D and let t be the type consisting of the following conditions: $a_n < x, x < b_n$ and $[d_n, x] = 0$, for all $n \in \mathbb{N}$. If to is any finite subset of t and $\varepsilon > 0$, then for a large enough n we have that a_n ε-approximately realizes t_0. By countable saturation of C, some $c \in C$ realizes t. This c satisfies the requirements of the AA-CRISP for a_n, b_n and D.

Recall that a C^*-algebra C is an SAW^*-algebra if any two σ-unital subalgebras A and B of C are orthogonal if and only if they are separated. By Theorem (2.2.26) the following is a strengthening of the result that every corona of a σ-unital C^*-algebra is an SAW^*-algebra ([127]). (By [127], CRISP implies SAW^* but we include a simple direct proof below.)

Proposition (2.2.12)[114]: Every countably degree-*1* saturated C*-algebra C is an SAW*-algebra.

Proof. Assume A and B are σ-unital subalgebras of C such that $ab = 0$ for all $a = A$ and all $b \in B$. Let a_n, for $n \in \mathbb{N}$ and a_n, for $n \in \mathbb{N}$, be an approximate identity of A and B, respectively. Consider type t_{AB} consisting of the following expressions, for all n.

(i) $a_n x = a_n$,

(ii) $x b_n = 0$

(iii) $x = x^*$.

Every finite subset of t_{AB} is ε-realized by a_n for a large enough n. If c realizes t_{AB}, then $ac = a$ for all $a \in A$ and $cb = 0$ for all $b \in B$. Moreover, c is self-adjoint by (iii) and $|c|$ still satisfies the above.

Assume B, C and D are subalgebras of a C^*-algebra M. We say that D derives B if for every $d \in D$ the derivation $\delta_d(x) = dx — xd$ maps B into itself. The following is an extension of Higson's formulation of Kasparov's Technical Theorem ([124], also [127]).

We say that a C^*-algebra M has KTT if the following holds: Assume A, B, and C are subalgebras of M such that $A \perp B$ and C derives B. Furthermore assume A and B are a-unital and C is separable. Then there is a positive element $d \in M$ such that $d \in C' \cap M$, the map $x \longmapsto xd$ is the identity on B, and the map $x \longmapsto dx$ annihilates A.

Proposition (2.2.13)[114]: Every countably degree-1 saturated C*-algebra has KTT.

Proof. Assume A, B and C are as above. Since B is a-unital we can fix a strictly positive element $b \in B$. Then $b^{1/n}$, for $n \in \mathbb{N}$, is an approximate unit for B. An easy computation demonstrates that for every $c \in C$ the commutators $[b1/n, c]$ strictly converge to 0 (see the first paragraph of the proof of Theorem 8.1 in [127]). They therefore converge to 0 weakly. The Hahn-Banach Theorem combined with the separability of C now shows that one can extract an approximate unit (e_m) for B in the convex closure of $\{b^{1/n} : n \in \mathbb{N}\}$ such that the commutators $[e_m, c]$ norm-converge to 0 for every $c \in C$.

In other words, B has an approximate unit (e_m) which is C-quasicentral. Fix a countable approximate unit (f_n) of A and a countable dense subset $\{c_m\}$ of C. Consider the type t consisting of the following conditions, for all m and all n.

$$\|e_n x - e_n\| = 0$$
$$\|x f_n\| = 0$$
$$\|[c_m, x]\| = 0$$
$$\|x � x^*\| = 0.$$

For every finite subset F of this type and every $\varepsilon > 0$ there exists an m large enough so that all the conditions in F are e-satisfied with $x = e_m$. Therefore the type t is consistent and by countable degree�1 saturation it is satisfied by some d_0. Then $d = |d_0|$ is as required.

A C^*-algebra M is sub-Stonean if for all b and c in M such that $bc = 0$ there are positive contractions f and g such that $bf = b, gc = c$ and $fg = 0$. By considering $B = C^*(b)$ and $C = C^*(c)$ and noting that B and C are orthogonal, one easily sees that every SAW^* algebra is sub-Stonean. The following strengthening was introduced by Kirchberg [125].

Definition (2.2.14)[114]: A C^*-algebra C is sub�σ�Stonean if for every separable sub-algebra A of C and all positive b and c in C such that $bAc = \{0\}$ there are contractions f and g in $A' \cap C$ such that $fg = 0, fb = b$ and $gc = c$.

The fact that for a separable C^*-algebra A the relative commutant of A in its ultrapower associated with a nonprincipal ultrafilter on ℕ (as well as the related algebra $F(A) = (A' \cap A^{\mathcal{U}})/Ann(A, A^{\mathcal{U}})$, see [125]) is sub-a Stonean was used in [125] to deduce many other properties of the relative commutant. Several proofs in [125], can easily be recast in the language of logic for metric structures.

Before we strengthen Kirchberg's result by proving countably degree-1 saturated algebras are sub-σ-Stonean (Proposition (2.2.16)) we show a lemma.

Lemma (2.2.15)[114]: Assume M is countably degree-*1* saturated and B is a separable subalgebra. If I is a (closed, two-sided) ideal of B then there is a contraction $fG\ M \cap B'$ such that $af = a$ for all $a \in I$.

If moreover $c \in M$ is such that $c = \{0\}$, then we can choose f *so* that $fc = 0$ and $fIc = \{0\}$.

Proof. Fix a countable dense subset a_n, for $n \in \mathbb{N}$, of I and a countable dense subset b_n, for $n \in \mathbb{N}$,on B. Consider type t consisting of the following conditions.

(i) $\|a_n x - a_n\| = 0$ for all $n \in \mathbb{N}$,
(ii) $\|b_n x - x b_n\| = 0$ for all $n \in \mathbb{N}$.
(iii) $xc = 0$, and
(iv) $x a_n c = 0$ for all $n \in \mathbb{N}$.

We prove that t is consistent, and moreover that it is finitely approximately satisfiable by a contraction. By [118] I has a B-quasicentral approximate unit e_n, for $n \in \mathbb{N}$, consisting of positive elements. Since $Bc = \{0\}$ we have $e_n c = 0$, as well as $e_n a_m c = 0$ for all m and all n. Therefore every finite fragment of t is arbitrarily well approximately satisfiable by e_n for all large enough n. By Lemma (2.2.6) (applied with $[r, s] = [0,1]$) and

saturation of M there is a contraction $f \in M$ that realizes t. Then $fa = a$ for all $a \in I, f \in B' \cap M, f\ Ac = \{0\}$, and $fc = 0$, as required.
Proposition (2.2.16)[114]: Every countably degree-*1* saturated, C*-algebra is sub-σ-Stonean.

Proof. Fix A, b and c as in Definition (2.2.14). By applying Lemma (2.2.15) find a contraction $f \in M \cap A'$ such that $bf = b, fc = 0$ and $fAc = \{0\}$. Now let $C = C^*(A, c)$ and let J be the ideal of C generated by c. By applying Lemma (2.2.15) again (with left and right sides switched) with c replaced by f we find a contraction $g \in M \cap A'$ such that $fg = 0$, and $gc = c$.

By Theorem (2.2.26) the following is a strengthening of the result that every derivation of a separable subalgebra of the corona of a a-unital C^*-algebra is inner ([127]).
Proposition (2.2.17)[114]: Assume C is a countably degree-*1* saturated C*-algebra and B is a separable subalgebra. Then every derivation δ of B is of the form δ_c for some $c \in C$.

Proof. Fix a countable dense subset B_0 of B. Consider the type t_δ consisting of following conditions, for $b \in B_0$.

(i) $\|\|\| xb - bx - \delta(b)\| = 0$.

By [28] this type is consistent and if c realizes it then $\delta(b) = \delta_c(b)$ for all $b \in B$.

[109] proved that the Continuum Hypothesis implies that the Calkin algebra has $2^{\aleph_1}$ outer automorphisms. Since $k < 2^K$ for all cardinals k, this conclusion implies that the Calkin algebra has outer automorphisms. A simpler proof of Phillips-Weaver's result was given in [8]. The proof of Theorem (2.2.21) below is in the spirit of [109], but instead of results about KK-theory it uses countable quantifier-free saturation.

Recall that the character density of a C^*-algebra is the smallest cardinality of a dense subset. The following remark refers to the full countable saturation in logic for countable structures, not considered in (cf. [104]). The standard back-and-forth method shows that a fully countably saturated C^*-algebra of character density $\aleph_1$ has $2^{\aleph_1}$ automorphisms. Therefore, the Continuum Hypothesis implies that M has $2^{\aleph_1}$ automorphisms whenever M is an ultrapower of a separable C^*-algebra, a relative commutant of a separable C^*-algebra in its ultrapower, or an algebra of the form $\prod_n A_n / \bigoplus_n A_n$ for a sequence of separable unital C^*-algebras A_n, for $n \in \mathbb{N}$. Since $\aleph_1$ is always less than $2^{\aleph_1}$, in this situation, the automorphism group is strictly larger than the group of inner automorphisms. These issues will be treated in an upcoming joint with David Sherman. In the following we show how to construct $2^{\aleph_1}$ automorphisms in a situation where the algebra is only quantifier-free saturated.

Before proceeding to prove Theorem (2.2.21) we note that every countably saturated metric structure of character density $\aleph_1$ has $2^{\aleph_1}$ automorphisms. We don't know whether the Continuum Hypothesis implies that every corona of a separable C^*-algebra has $2^{\aleph_1}$ automorphisms (but see [118]).
By Theorem (2.2.21) and Theorem (2.2.23) we have the following:

Corollary (2.2.18)[114]: Assume the Continuum Hypothesis. Assume A is a C*- algebra such that for every separable subalgebra B of $M(A)$ there is a B-quasi- central approximate unit for A consisting of projections and the center of $C(A)$ is separable. Then $C(A)$ has $2^{\aleph_1}$ outer automorphisms.

Recall that an automorphism Φ of a C^*-algebra C is approximately inner if for every $\varepsilon > 0$ and every finite set F, there is a unitary u such that $\|\Phi(a) - uau^*\| < \varepsilon$ for all $a \in F$. An approximately inner $*$ isomorphism from a subalgebra of C into C is defined analogously.

Lemma (2.2.19)[114]: Assume C is a countably quantifier-free saturated C*-algebra and B is its separable subalgebra. If $\Phi: B \to C$ is an approximately inner $*$isomorphism then there is a unitary $u \in C$ such that $\Phi(b) = ubu^*$ for all $b \in B$.

Proof. This is essentially a consequence of Lemma (2.2.7) Fix a countable dense subset B_0 of B. Consider the type t_{Φ}consisting of all conditions of the form $\|xbx^* - \Phi(b)\| = 0$ for $b \in B_0$together with $xx^* = 1$ and $x^*x = 1$. The assumption that Φ is approximately inner is equivalent to the assertion that $t_{\Phi}$is consistent. Since B_0 is countable, by countable quantifier-free saturation there exists $u \in C(A)$ that realizes t_{Φ}**.** Such u is a unitary which implements Φ.

Lemma (2.2.20)[114]: Assume C is a countably quantifier-free saturated, simple C^*-algebra whose center is separable. If Φ is an automorphism of C and A is a separable subalgebra of C then there is an automorphism $\Phi'$of C distinct from Φ whose restriction to A is identical to the restriction of Φ to A. Moreover, if Φ is inner then Φ'can be chosen to be inner.

Theorem (2.2.21)[114]: If C is a countably quantifier-free saturated C*-algebra of character density $\aleph_1$ whose center is separable then C has $2^{\aleph_1}$automorphisms.

Proof. By using Lemma (2.2.19) and Lemma (2.2.20) we can construct a complete binary tree of height $\aleph_1$ whose branches correspond to distinct automorphisms. This standard construction is similar to the one given in [109] but much easier, since in our case the limit stages are covered by Lemma (2.2.19), and in [109] most of the effort was made in the limit stages.

The strict topology on $M(A)$ is the topology induced by the family of seminorms $\|(x-y)a\|$, where a ranges over A. If A is separable then the strict topology on $M(A)$ has a compatible metric,$\|(x-y)a\|$, where a is any strictly positive element of A.

We note that for any sequence of C^*-algebras A_n, for $n \in \mathbb{N}$, the algebra $\prod_n A_n/\bigoplus_n A_n$ is fully countably saturated. This is a straightforward analogue of a well-known result in classical model theory (cf. [104], [116]).

The proof of Theorem (2.2.5) is a warmup for the proof of Theorem (2.2.26). In Proposition (2.2.28) we shall see that the conclusion of Theorem (2.2.23) does not follow from the assumptions of Theorem (2.2.26) Let us start by recalling the statement of Theorem (2.2.5)

We shall write $\bar{b}$ for an n-tuple, hence

$$\bar{b} = (b_1, \ldots, b_n)$$

in order to simplify the notation. We also write

$$q\bar{b} = (qb_1,\dots,qb_n).$$

In our proof of Theorem (2.2.23) we shall need the following fact.

Lemma (2.2.22)[114]: Assume $P(x_1,\dots,x_n)$ is a^*-polynomial with coefficients in $a\ C^*$-algebra C. Then there is constant $K < \infty$, depending only on P, such that for all a and $b_1,\dots,b_n$ in C we have

$$\|[a, P(\bar{b})]\| \le K \max_c \|[a,c]\| \, \|a\| \max_{j\le n} \|bj\|$$

where c ranges over coefficients of Pand $b_1,\dots,b_n$.

If in addition q is a projection then we have

$$\|qP(\bar{b}) - qP(q\bar{b})q\| \le K \max_c \|[q,c]\| \, \|a\| \max_{j\le n} \|b_j\|.$$

Proof. The existence of constant K satisfying the first inequality can be proved by a straightforward induction on the complexity of P. For the second inequality use the first one and the fact that $q = q^{d+1}$, where d is the degree of P in order to find a large enough K.

Theorem (2.2.23)[114]: Assume A is $a\ \sigma$-unital C^*-algebra such that for every separable subalgebra B of $M(A)$ there is a B-quasicentral approximate unit for A consisting of projections. Then its corona $C(A)$ is countably quantifier-free saturated.

Proof. Fix a countable quantifier-free type t over $C(A)$ and enumerate all polynomials occurring in it as $P_n(\bar{x})$, for $n \in \mathbb{N}$. By re-enumerating and adding redundancies we may assume that all variables of P_n are among $x_1,\dots,x_n$. Let $P_n^0(\bar{x})$ be a polynomial over M(A) corresponding to $P_n(x)$. Let K_n be a constant corresponding to P_n^0 as given by Lemma (2.2.22) Let B be a separable subalgebra of $M(A)$ such that all coefficients of all polynomials $P_n^0(\bar{x})$ belong to B.

Let r_n for $n \in \mathbb{N}$ be such that t is the set of conditions $\|P_n(\bar{x})\| = r_n$ for $n \in \mathbb{N}$. For all n fix $b_1^n,\dots,b_n^n$ such that

$$\left|\left\|\pi\left(P_j^0(b_1^n,\dots,b_n^n)\right)\right\| - r_n\right| < 2^{-n}$$

for all j $\le n$ and $\|b_k^n\| \le 2$. The latter is possible by our assumption that the condition $\|x_n\| \le 1$ belongs to t for all k.

Let q_n, for $n \in \mathbb{N}$, be a B-quasicentral approximate unit for A consisting of projections. By going to a subsequence we may assume the following apply for all $j \le n$ (with $q_0 = 0$):

(i) $\|[q_n, a]\| < 2^{-n}K_n^{-1}$ when a ranges over coefficients of P_j^0 and all $b_1^j,\dots,b_j^j$,

(ii) $\left|\|(q_{n+1} - q_n)P_j^0(b_1^j,\dots,b_j^j)(q_{n+1} - q_n)\| - r_n\right| < 1/n$,

Let

$$P_n = q_{n+1} - q_n$$

For every k the series $\sum_n P_n b_k^n P_n$ is convergent with respect to the strict topology. Let b_k be equal to the sum of this series. By the second inequality of Lemma (2.2.22) and (i) we have that for all $k \le n$

(iii) $\left|\left\|p_n\, P_k^0(b_1,\dots,b_k)\right\| - \|p_n\, P^0\,(P_nb_1P_n,\dots,P_nb_kP_n)P_n\|\right| < 2^{-n}$.

Since $p_nb_kp_n = p_nb_k^np_n$, we conclude that

$$\left\|P_k\left(\pi(\bar{b})\right)\right\| = \left\|\pi\left(P_j^0(\bar{b})\right)\right\| = \lim_n sup\,\|P_nP_j^0(P_nb_1^nP_n,\dots,P_nb_j^nP_n)\| = r_n$$

Therefore $\pi(b_n)$, for $n \in \mathbb{N}$, realizes t in $C(A)$

We shall use [127] which states that if $0 \le a \le 1$ and $\|b\| = 1$, then $\|[a,b]\| \le \varepsilon \le 1/4$ implies $\left\|[a^{1/2},b]\right\| \le 5\varepsilon^{1/2}/4$. We shall also need the following lemma.

Lemma (2.2.24)[114]: Assume a and b are positive operators. Then $\|a + \mathrm{b}\| \ge \max(\|\mathrm{a}\|,\|\mathrm{b}\|)$.

Proof. We may assume $1 = \|a\| \ge \|b\|$. Fix $\varepsilon > 0$ and let ξ be a unit vector such that $\eta = \xi - a\xi$ satisfies $\|\eta\| < \varepsilon$ .Then $Re(a\xi|b\xi) = Re(\xi|b\xi) + Re(\eta|b\xi) \ge Re(\eta|b\xi) > -\varepsilon$ since $b \ge 0$. We therefore have

$$\|(a + b)\xi\|^2 = ((a + b)\xi|(a + b)\xi)$$

$$= \|a\xi\|^2 + \|b\xi\|^2 + 2Re(a\xi|b\xi) > 1 + \|b\xi\|^2 - 2\varepsilon$$

and since $\varepsilon > 0$ was arbitrary the conclusion follows.

Lemma (2.2.25)[114]: Assume M is a C^*-algebra and a σ-unital C^*-algebra A is an essential ideal of M. Furthermore assume F_n, for $n \in \mathbb{N}$, is an increasing sequence of finite subsets of the unit ball of M and ε_n, for $n \in \mathbb{N}$, is a decreasing sequence of positive numbers converging to 0. Then A has an approximate unit e_n, for $n \in \mathbb{N}$ such that with (setting $e_{-1} = 0$)

$$f_n = (e_{n+1} - e_n)^{1\backslash 2}$$

for all n and all $a \in F_n$ we have the following:

(iv) $\|[a,f_n]\| \le \varepsilon_n$,

(v) $\|f_naf_n\| \ge \|\pi(a)\| - \varepsilon_n$ (where $\pi\colon M \to M/A$ is the quotient map),

(vi) $\|f_mf_n\| = 0$ if $|m - n| \ge 2$,

(vii) $\|[f_n,f_{n+1}]\| \le \varepsilon n$.

Proof. In order to take care of the condition(vi)we do the following. Let h be a strictly positive element of A. By continuous functional calculus we choose an approximate unit (e_n^{-1}) of A satisfying (vi).

Let $\delta_n = (4\varepsilon_n/25)^2$. By [118] inside the convex closure this approximate unit we can find another approximate unit$(e_n^0)$of A such that

(viii) $\|e_n^0a - ae_n^0\| \le \delta_n$ for all $a \in F_n \cup \{e_n^0 : i < n\}$.

We can moreover assure that there is an increasing sequence of natural numbers $m(n)$, for $n \in \mathbb{N}$, such that e_n^0 is in the convex closure of $\{e_k^{-1} : m(n) \le k < m(n + 1)\}$. This will assure every subsequence (e_n) of (e_n^0) satisfies (vi).

For such a subsequence $(e_n)$and $f_n$defined as above we will have (iv) and (vii) by the choice of δ_n and [127]. Since A is an essential ideal of M, there is a faithful representation $\alpha\colon M \to B(H)$such that $\alpha\,[A]$ is an essential ideal of $B(H)$ (this is essentially by [100]). In particular $\alpha(e_n)$ strongly converges to 1_H. Therefore for every $a \in M$, $m \in \mathbb{N}$, and $\varepsilon > 0$ there is n large enough so that$\left\|\alpha\left(a(e_n - e_m)\right)\right\| \ge \|\alpha(a)\| - \varepsilon$. Using this observation we can recursively find a subsequence (e_n) of (e_n^0) such that $\|(e_{n+1} - e_n)a\| \geq \|\pi(a)\| - \delta_n$ for all $a \in F_n$.Therefore $\|f_n a f_n\| \geq \|\pi(a)\| - \varepsilon_n$ for all $a \in F_n$ and (v) holds.

Fix a σ-unital C^*-algebra A; let $M = M(A)$, and $\varepsilon_n = 2^{-n}$. Now by applying Lemma(2.2.25) we get $A, M, F_n, (\mathrm{e_n})$ and (f_n), for $n \in \mathbb{N}$. We shall show that in this situation these objects have the additional properties in formulas (ix)-(xvi) below.

(ix) The series $\sum_n f_n^2$ strictly converges to 1.

Since A is c-unital, we can pick a strictly positive $a \in A$. Therefore the strict topology is given by compatible metric $d(b,c) = \|a(b - c)\|$. Fix $\varepsilon > 0$. Let n be large enough so that $\|ae_{n+1} - a\| < \varepsilon$. Since $1 - e_{n+1} = \sum_{j=n+1}^{\infty} f_j^2$, (ix) follows.

(x) For every sequence (b_j) in the unit ball of M the series $\sum_j f_j\, b_j\, f_j$ is strictly convergent.

We first note that $0 \leq c \leq d$ implies $\|cb\| \leq \|db\|$for all b. This is because $\|cb\|^2 = \|b^* c^2 b\| \leq \|b^* d^2 b\| = \|db\|^2$.

Since every element b of a C^*-algebra is a linear combination of four positive elements $b = c_0 - c_1 + ic_2 - ic_3$, we may assume $b_j \geq 0$ for all j. Fix $\varepsilon > 0$ and find n large enough so that (with $a \in \mathrm{A}$ strictly positive) $\left\|\sum_{j=n}^{\infty}(f_j^2)a\right\| < \varepsilon$. Then $0 < \sum_{j\geq n} f_j\, b_j\, f_j \leq \sum_{j\geq n} f_j^2$. Therefore by the above inequality applied with $c = \sum_{j\geq n} f_j\, b_j\, f_j$ and $d = \sum_{j\geq n} f_j^2$ we have $\|ca\| \leq \|da\| \leq \varepsilon$.

(xi) $\left\|\sum_j f_j x_j f_j\right\| < sup_j \left\|f_j x_j f_j\right\|$ for every norm-bounded sequence (x_j).
(xii) If in addition $\sup_j \left\|f_j x_j f_j\right\| = \sup_j \left\|x_j\right\|$ then we moreover have the equality in (xi).

In order to prove (xi) consider the C^*-algebra $N = \prod_{\mathbb{N}} M$. Each map

$$N \ni (x_k)_{k\in\mathbb{N}} \mapsto f_j\, x_j\, f_j \in M$$

for j $\in \mathbb{N}$ is completely positive on N, and therefore for each n $\in \mathbb{N}$ the map $(x_k)_{k\in\mathbb{N}} \mapsto \sum_{\mathrm{j\leq n}} \mathrm{f_j} x_\mathrm{j} \mathrm{f_j}$is completely positive as well. The supremum of these maps is also a completely positive map. By the assumption that $\sum_j f_j^2 = 1$ this map is also unital, and therefore of norm 1. The inequality (xi) follows.

In order to prove (xii) $let\ \alpha = sup_j \left\|x_j\right\|$. We may assume $\alpha = 1$. Fix $\varepsilon > 0$, unit vector ξ, and n such that $\|(\mathrm{f_n x_n f_n})\xi\| > 1 - \varepsilon$. Then $\|\mathrm{f_n}\xi\| \geq 1 - \varepsilon$ and therefore $|(\mathrm{f_n^2}\xi|\xi)| = \|\mathrm{f_n}\xi\| \geq 1 - \varepsilon$ and this implies that $\|\xi - \mathrm{f_n^2}\xi\| \leq \varepsilon$. Since $\sum_j f_j^2 = 1$, this shows that $\left\|\sum_j(\mathrm{f_j} x_\mathrm{j} \mathrm{f_j})\xi\right\| \approx \|(\mathrm{f_n} x_\mathrm{n} \mathrm{f_n})\xi\|$ and the conclusion follows.

Recall that $\pi: M(A) \to C(A)$ is the quotient map. In the following the norm on the left-hand side of the equality is computed in the corona and the norm on the right-hand side is computed in the multiplier algebra.

(xiii) $\left\|\pi\left(\sum_j \mathrm{f_j} x_\mathrm{j} \mathrm{f_j}\right)\right\| = \mathrm{limsup_j}\left\|\mathrm{f_j} x_\mathrm{j} \mathrm{f_j}\right\|$ for every bounded sequence (x_j) such

that $\sup_j \left\|\mathrm{f_j} x_\mathrm{j} \mathrm{f_j}\right\| = \sup_j \left\|x_\mathrm{j}\right\|$.

Since $\sum_{j=0}^{\infty} \mathrm{f_j} x_\mathrm{j} \mathrm{f_j} - \sum_{j=m}^{\infty} \mathrm{f_j} x_\mathrm{j} \mathrm{f_j}$ is in A for all $m \in \mathbb{N}$, the inequality $\leq$ follows from (xi) and $\|\pi(a)\| \leq \|a\|$. Similarly, $\geq$ follows from (xii).

The converse inequality follows by Lemma (2.2.24)

(xiv) $X_{(f_n)} = \{a \in M: \sum_n \|[a, f_n]\| < \infty\}$is a subalgebra of M including $C^*(U_n\, F_n)$.

Since $b \in F_j$ implies $\|[b, f_n]\| \leq 2^{-n}$ for all $n \geq j$, we have $\mathrm{U_j\ F_j} \subseteq X_{(f_n)}$.

For a and b in M we have $[a + b, f_n] = [a, f_n] + [b, f_n]$, $\|[a^*, f_n]\| = \|[a, f_n]\|$ and $\|[ab, f_n]\| \leq \|a\|.\ \|[b, f_n]\| + \|b\|\ .\ \|[a, f_n]\|$. Therefore $X_{(f_n)}$ is a $*$subalgebra of M. $X_{(f_n)}$ is not necessarily norm-closed but this will be of no consequence.

(xv) The map $\Lambda = \Lambda_{(f_n)}$ from M into M defined by

$$\Lambda(a) = \sum\nolimits_n f_n a f_n$$

is completely positive and it satisfies $b - \Lambda(b) \in A$ for all $b \in X_{(f_n)}$.

Note that $\|\Lambda(b)\| \leq \|b\|$ by (xi), and the map is clearly completely positive. Fix $b \in X_{(f_n)}$ and $\varepsilon > 0$. Since $b \in X_{(f_n)}$ the series $\delta_j = \|f_j b - b f_j\|$ is convergent, and we can pick n large enough to have $\sum_{j\geq n}\|f_j b - b f_j\| \leq e$. We write $c \sim_A d$ for $c - d \in A$ and $c \sim_\varepsilon d$ for $\|c - d\| \leq \varepsilon$ (clearly the latter is not an equivalence relation). We have $\sum_{\boldsymbol{j\leq n}} f_j b f_j \in A$. Also, with $\delta = \sum_{j\geq n} \delta_j$ we have

$$(1 - e_n)b = \sum\nolimits_{j=n}^{\infty} f_j^2 b \sim_\delta \sum\nolimits_{j=n}^{\infty} f_j b f_j$$

and the conclusion follows.

(xvi) If $sup_j \|x_j\| < \infty$ and $\delta_j = sup_{i\geq j} \|[x_j, f_i]\|$ are such that $\sum_j \delta_j < \infty$, then $x = \sum_j f_j x_j f_j$ belongs to $X_{(f_n)}$.

We have$f_n \left(\sum_j f_j x_j f_j\right) = f_n\ (\sum_{\boldsymbol{j=n-1}}^{\boldsymbol{n+1}} f_j x_j f_j)$. Since $\|f_k, f_{k+1}\| \leq \varepsilon_k$ we have

$$\|x, f_n\| \leq \sum\nolimits_{j=n-1}^{n+1} \|\mathrm{f_j x_j f_j, f_n}\| \leq 4 sup_j \|x_j\| \epsilon_{n-1} + \delta_{n-1} + \delta_n + \delta_{n+1}$$

and the conclusion follows.

Theorem (2.2.26)[114]: If A is a σunital C*-algebra then its corona $C(A)$ is countably degree-*1* saturated.

Proof. Fix a σ-unital algebra A and let $\pi: M(A) \to C(A)$ be the quotient map.

Fix degree-1 $*$polynomials $P_n(\bar{x})$ with coefficients in $C(A)$ and compact subsets $K_n \subseteq \mathbb{R}$ such that for every n the system

(xvii) $\|P_j(\bar{x})\| \in (K_j)_{\boldsymbol{1/n}}$ for all $j \leq n$

has a solution in $C(\,A)$. Without a loss of generality all the inequalities of the form $\|x_n\| \leq 1$, for $n \in \mathbb{N}$, are in the system. By compactness, we can assume each K_n is a singleton

$\{r_n\}$. Therefore we may assume (xvii) consists of conditions of the form $|\|P_n(\bar{x})\| - r_n| \leq 1/m$, for all m and n. By re-enumerating $P_n's$ and adding redundancies, we may also assume that only the variables x_j, for $j \leq n$, occur in P_n for every n. For each m fix an approximate solution $\dot{x}_j\ (m) = \pi\left(x_{\ j}(m)\right)$, for $j \leq m$, as in (xvii). Therefore

(xviii) $\left\|P_k\left(\pi\left(\bar{x}(m)\right)\right)\right\| - r_k| \leq 1/m$ for all $k \leq m$.

We choose all $x_k\ (m)$ to have norm ≤ 1.

Let $P_n^0(\bar{x})$ be a polynomial with coefficients in $M(\mathrm{A})$ that lift to the corresponding coefficients of $P_n(\bar{x})$. Let F_n be a finite subset of $M(A)$ such that $\pi(F_n)$ includes the following:

(i) all coefficients of every P_j^0 for $j \leq n$,

(ii) $\{x_k\ (m) : k \leq m\}$ satisfying (xviii) for all $m \leq n$, and

(iii) $\{\mathrm{P}_j^0\left(x_0(j), \ldots, x_j\ (j)\right) : j \leq n\}$.

With $\varepsilon_n = 2^{-n}$ let (e_n) and (f_n) be as guaranteed by Lemma (2.2.25).Since $\left\|x_j(i)\right\| \leq 1$, by (x) we have that

$$\mathrm{y_i} = \sum_{\mathrm{i}} f_j \mathrm{x_i(j)f_i}$$

belongs to $M(A)$for all i, and (xvi) implies $y_i \in X_{(f_n)}$ for all i.

We shall prove $\left\|\left(P_n\pi(\bar{y})\right)\right\| = r_n$ for all n.

By (xi) we have $\|y_i\| \leq 2$. Fix n and a monomial ax_kb of $P_n^0(\bar{x})$. Then for all $j \geq n$ we have

$$\|af_ix_k(j)f_ib - f_iax_k(j)bf_i\| \leq \varepsilon_j.(|a| + |b|)$$

and therefore the sum of these differences is a convergent series in A and we have

(xix) $a\left(\sum_j f_ix_k(j)f_i\right)b \sim_A \sum_j (f_iax_k(j)bf_i)$.

Since the polynomial $P_n^0(\bar{x})$ has degree 1, all of its nonconstant monomials are either of the form ax_kb or of the form ax_k^*b for some k, a and b, and by (xix) (writing $\sum_j f_i\bar{y}f_i$ for the $n+1$- tuple $\left(\sum_j f_iy_0f_i, \ldots \sum_j f_iy_nf_i\right)$

$$P_n^0\left(\sum_j f_iy_kf_i\right) \sim_A \sum_j f_iP_n^0(\bar{y})f_i\,.$$

By (xv) we have $\sum_j f_iy_if_i \sim_A \sum_j f_iy_if_i$ for all i and therefore

$$P_n^0(\bar{y}) \sim_A P_n^0\left(\sum_j f_i\bar{y}f_i\right) \sim_A \sum_j f_iP_n^0(\bar{y})f_i\,.$$

Using this, by (xii) we have that

$$\|P_n(\pi(\bar{y}))\| = \|\pi(P_n^0(\bar{y}))\| = \limsup_j \|f_i P_n^0(\bar{y}) f_j\| = r_n.$$

Therefore $\pi(\bar{y})$ is a solution to the system. Since the inequality was in the system for all k we also have $\|y_k\| \leq 1$ for all k and this concludes$\|x_k\| \leq 1$ was proof.

We prove that the Calkin algebra is not countably saturated (cf. [104]). In Proposition (2.2.27) we construct a consistent type consisting of universal formulas that is not realized in the Calkin algebra. In Proposition (2.2.28) we go a step further and present a proof, due to N. Christopher Phillips, that some consistent quantifier-free type is not realized in the Calkin algebra.

For a unitary u in a C^*algebra A let

$$\xi(u) = \{j \in \mathbb{N} \mid u \text{ has } a\ j\text{th root}\}.$$

By Atkinson's Theorem, every invertible operator in the Calkin algebra is the image of a Fredholm operator in $\mathcal{B}(H)$ and therefore $\xi(u)$ is either $\mathbb{N}$ or $\{j \mid j \text{ divides } m\}$ for some $m \in \mathbb{N}$, depending on whether the Fredholm index of u is 0 or $\pm m$.

Recall that a supernatural number is a formal expression of the form $\prod_i p_{p_i}^{k_i}$, where$\{p_i\}$I s the enumeration of primes and each k_i is a natural number (possibly zero) or ∞. The divisibility relation on supernatural numbers is defined in the natural way.

Proposition (2.2.27)[114]: For any supernatural number n the type $t(n)$ consisting of following conditions is approximately finitely satisfiable, but not realizable, in the Calkin algebra.

(I) $x_o x_0^* = 1, x_0^* x_o = 1,$
(ii) $x_k^k = x_o,$ whenever k is a natural number that divides n,
(iii) $\inf_{||y||=1} \left\| y^k - x_o \right\| \geq 1,$ whenever k is a natural number that does not divide n.

In particular, the Calkin algebra is not countably saturated.

Proof. We have $n = \prod_j p_j^{k_j}$, where $(p_j)$is the increasing enumeration of primes and $k_j \in \mathbb{N} \cup \{\infty\}$.

Let s denote the unilateral shift on the underlying Hilbert space H and let $\dot{s}$ be its image in the Calkin algebra. For $l \in \mathbb{N}\ let\ n_l = \prod_{j=1}^{l} P_j$. We claim that

$$\xi(\dot{s}^{nl}) = [m \in \mathbb{N} \mid m \text{ divides } n\}.$$

The inclusion is trivial. In order to prove the converse inclusion fix $k \in \mathbb{N}$ that does not divide n_l. Assume for a moment that $\dot{s}^{nl}$ has a k-th root $\dot{v}$ in $C(H)^{\mathcal{U}}$.Let u and w be elements of $\mathcal{B}(H)$ mapped to $\dot{s}^{nl}$ and $\dot{v}^k$ by the quotient map.Then they are Fredholm operators with different Fredholm indices and $\|\pi(u)\| = \|\pi(w)\| = 1$. Essentially by [108] we have $\left||\pi(u - w)\right|| \geq 1$,and therefore $v = n(w)$ is not k-th root of $\dot{s}^{nl}$.

Proposition (2.2.28) below was communicated to us by **N**. Christopher Phillips in [128]. While the proof in [128] relied entirely on known results about Pext and a topology on Ext (more precisely, [129], [118], [130], and [131]).

Proposition (2.2.28)[114]: There is a countable degree-1 type over the Calkin algebra that is approximately finitely realizable by unitaries but not realizable by a unitary. In particular, the Calkin algebra is not countably quantifier-free saturated.

Proof. We include more details than a C^*-algebraist may want to see. Recall that for a C^*- algebra A the abelian semigroup $Ext(A)$ is defined as follows: On the set of $*$-homomorphisms $\pi: A \to C(H)$ consider the conjugacy relation by unitaries in $C(H)$. On the set of conjugacy classes define addition by letting $\pi_1 \oplus \pi_2$ be the direct sum, where $C(H)$ is identified with $C(H \oplus H)$. The only fact about Ext that we shall need is that there exists a simple separable C^*-algebra A such that A is a direct limit of algebras whose Ext is trivial, but $Ext(A)$ is not trivial. For example, the CAR algebra has this property and we shall sketch a proof of this well-known fact below.

Now fix A as above and let $\pi_1: A \to C(H)$ and $\pi_2 : A \to C(H)$ be inequivalent $*$-homomorphisms. Since A is simple both π_1 and π_2 are injective and $F(\pi_1(a)) = \pi_2(a)$ defines a map F from $\pi_1[A]$ to $\pi_2[A]$. This map is not implemented by a unitary, but if $A = lim_n A_n$ so that $Ext\,(A_n)$ is trivial for every n, then the restriction of F to $\pi_1[A_n]$ is implemented by a unitary. Fix a countable dense subset D of $\pi_1[A_n]$. Then the countable degree-1 type t consisting of all conditions of the form $xa = F(a)x$, for $x \in D$, is approximately finitely realizable by a unitary, but not realizable by a unitary.

We now sketch a proof that Ext of the CAR algebra $A = \otimes_n M_2(\mathbb{C})$ is nontrivial. Write A as a direct limit of $M_{2^n}(\mathbb{C})$ for $n \in \mathbb{N}$. While $Ext(M_{2^n}(\mathbb{C}))$ is trivial, the so-called strong Ext of $M_{2^n}(\mathbb{C})$ is not. Two $*$- homomorphisms of $M_{2^n}(\mathbb{C})$ into $C(H)$ are strongly equivalent if they are conjugate by $\dot{u}$, for aunitary $u \in B(H)$. Every unital $*$- homomorphism Φ of $M_{2^n}(\mathbb{C})$ into $C(H)$ is lifted by a $*$- homomorphism Φ_0 into $\mathcal{B}(H)$and the strong equivalence class of Φ is uniquely determined by the codimension of $\Phi_0(1)$ modulo 2^n. Any unitary u in $C(H)$ that witnesses such Φ is conjugate to the trivial representation of $M_{2^n}(\mathbb{C})$which necessarily has Fredholm index equal to the codimension of $\Phi_0(1)$ modulo 2^n. Now write M_{2^∞} as $\otimes_{\mathbb{N}} A_n$ where $A_n \cong M_2(\mathbb{C})$ for all n. Recursively find *-homomorphisms π_1^n and π_2^n from $\otimes_{j \le n} A_j$ into the Calkin algebra so that (i)π_j^{n+1} extends π_j^n for all n and $j = 1, 2$, (ii) each π_1^n has trivial strong Ext class, and (iii) each π_2^n has strong Ext class 2^{n-1} (modulo 2^n). The construction is straightforward. The limits π_1 and π_2 are $*$-homomorphisms of the CAR algebra into the Calkin algebra such that the first one lifts to a homomorphism of the CAR algebra into $\mathcal{B}(H)$ and the other one does not.

Corollary (2.2.29)[370]: Assume M is countably degree-*1* saturated and B is a separable subalgebra. If I is a (closed, two-sided) ideal of B then there is a contraction f_j G $M \cap B'$ such that $a^j f_j = a^j$ for all $a^j \in I$.

If moreover $c \in M$ is such that $c = \{0\}$, then we can choose f_j *so* that $f_j c = 0$ and $f_j I c = \{0\}$.

Proof. Fix a countable dense subset a_n^j, for $n \in \mathbb{N}$, of I and a countable dense subset b_n^j, for $n \in \mathbb{N}$,on B. Consider type t consisting of the following conditions.

(i) $\sum_j \|a_n^j x - a_n^j\| = 0$ for all $n \in \mathbb{N}$,
(ii) $\sum_j \|b_n^j x - x b_n^j\| = 0$ for all $n \in \mathbb{N}$.
(iii) $xc = 0$, and
(iv) $\sum_j x a_n^j c = 0$ for all $n \in \mathbb{N}$.

We prove that t is consistent, and moreover that it is finitely approximately satisfiable by a contraction. By [118] I has a B-quasicentral approximate unit e_n, for $n \in \mathbb{N}$, consisting of positive elements. Since $Bc = \{0\}$ we have $e_n c = 0$, as well as $e_n a_m^j c = 0$ for all m and all n. Therefore every finite fragment of t is arbitrarily well approximately satisfiable by e_n for all large enough n. By Lemma (2.2.6) (applied with $[r, s] = [0,1]$) and saturation of M there is a contraction $f_j \in M$ that realizes t. Then $f_j a^j = a^j$ for all $a^j \in I, f_j \in B' \cap M, \sum_j f_j Ac = \{0\}$, and $\sum_j f_j c = 0$, as required.

Corollary (2.2.30)[370]: Assume a and $a + \epsilon$ are positive operators. Then $\|a + \text{b}\| \geq \max(\|\text{a}\|, \|\text{a} + \varepsilon\|)$.

Proof. We may assume $1 = \|a\| \geq \|a + \epsilon\|$. Fix $\varepsilon > 0$ and let ξ be a unit vector such that $\eta = \xi - a\xi$ satisfies $\|\eta\| < \varepsilon$ .Then $Re(a\xi|(a + \epsilon)\xi) = Re(\xi|(a + \epsilon)\xi) + Re(\eta|(a + \epsilon)\xi) \geq Re(\eta|(a + \epsilon)\xi) > -\varepsilon$ since $a + \epsilon \geq 0$. We therefore have

$$\|(2a + \epsilon)\xi\|^2 = ((2a + \epsilon)\xi|(2a + \epsilon)\xi)$$

$$= \|a\xi\|^2 + \|(a + \epsilon)\xi\|^2 + 2Re(a\xi|(a + \epsilon)\xi) > 1 + \|(a + \epsilon)\xi\|^2 - 2\varepsilon$$

and since $\varepsilon > 0$ was arbitrary the conclusion follows.

Section (2.3): Certain C** Algebras Which are Coronas of Banach Algebras

The study of the commutant modulo the Hilbert–Schmidt class of a normal operator with rich spectrum ([140], [133]) has shown that this Banach algebra together with its ideal of compact operators resembles in many ways the pair consisting of the algebra $\mathcal{B}(\mathcal{H})$ of all operators on a Hilbert space $\mathcal{H}$ and the ideal $\mathcal{K}(\mathcal{H})$of compact operators and that the analog of the Calkin algebra is also a C^* algebra. The purpose is to develop this analogy. We go beyond the case of a normal operator [140] or of a commuting n-tuple of hermitian operators [133] and deal with a general non-commuting n-tuple of operators and its commutant modulo a normed ideal which satisfies a certain quasicentral approximate unit condition relative to the n-tuple. The main result we obtain is that countable degree-1 saturation, in the model theory of ([114]), holds for the analog of the Calkin algebra, which is still a C^*algebra. We will refer to countable degree-1 saturation simply as "degree-1 saturation", for the sake of brevity. This adds to the list of nice properties of these analogs of the Calkin algebra and also adds to the list of C^*algebras satisfying degree -1 saturation ([114]). We also obtain a few other results. The existence of quasicentral approximate units for the ideal of compact operators in the Banach algebra we consider, as well as generalizations of some of the multiplier and duality results in [140].

The Calkin algebra which we obtain, Give hope that these algebras may be a good place to apply extensions of bi-Variant K-theory beyond KC^* −algebras ([136]) and cyclic cohomology ([135]).

We recall certain basic facts about normed ideals of compact operators ([137], [139]) and about the invariant $\mathcal{K}_y(\mathcal{J})$ where $\mathcal{J}$ is a normed ideal and τ an n-tuple of operators, which we used in the work on normed ideal perturbations of Hilbert space operators

([142],[14], [144]).

The main result is the existence of quasicentral approximate units for the compact ideal of the Banach algebras we study.

The construction we use has some of the flavor of the tridiagonal con-struction we used in the original proof of the non-commutative Weyl–von Neumann theorem [143], before the concept of quasicentral approximate units was abstracted ([7], [83]).
The fact that the analogue of the Calkin algebra is a C^*algebra.

We give the countable degree -1 saturation for the analogue of the Calkin algebra. The proof is along similar lines to those of the proof for coronas of C^*-algebras of Farah and Hart ([114]) with the added technical di culties arising from Banach algebra norms which don't allow continuous functional calculus. We were helped by the fact that in the case of the Calkin algebra the main technical lemma and the glueing construction simplify and becomes of the tridiagonal construction and the kind of approximately commuting partition of unity used to glue parts of operators in [143].

We deal with generalizations of multiplier and duality results from [140] to the general setting. Here once appropriate assumptions are found, the proofs in [140] generalize immediately.

The term normed ideal will be used as an Abbreviation for symmetrically normed ideal ([137], [139]) of compact operators on a separable infinite dimensional complex Hilbert space $\mathcal{H}$.

This is an ideal $0 \neq \mathcal{J} \subset \mathcal{B}(\mathcal{H})$ of the algebra of all bounded operators on $\mathcal{H}$ which is contained in $\mathcal{K}(\mathcal{H})$ the ideal of compact operators and which is endowed with a certain norm $|\ |_{\mathcal{J}}$with respect to which is a Banach space.The norm is given by$|T|_{\mathcal{J}} = |T|_{\Phi} = \Phi(s_1(T), s_2(T), \dots)$ where Φ is a norming function (see §3 in [137]) and $s_1(T) \geq s_2(T) \geq \dots$are the s-numbers of T . Given a norming function Φ we will use the notation in |9| and dente by (S_Φ , $|\ |_\Phi$ and $S_\Phi^{(0)}, |\ |_\Phi$) the normed ideals which are the set of all compact operators T so that $|T|_\Phi < \infty$ and, respectively, the closure in S_Φ of $\mathfrak{R}(\mathcal{H})$the ideal of finite rank operators.

We will always leave out $\mathcal{K}(\mathcal{H})$as a normed ideal.
If $(\mathcal{J}, |\ \ |_{\mathcal{J}})$ is a normed ideal we shall also use the notation for the closure of $\mathfrak{R}(\mathcal{H})$ in $\mathcal{J}$. Remark that since $|\ |_{\mathcal{J}} = |\ \ |_\Phi$ for some norming function $\Phi, \mathcal{J}^{(0)} = S_\Phi^{(0)}$ .Also if $|\ \ |_{\mathcal{J}} = |\ \ |_\Phi$we clearly have $S_\Phi^{(0)} \subset \mathcal{J} \subset s_\Phi$ and if $S_\Phi^{(0)} = s_\Phi$ the function Φ is called "mononorming.
If $\tau = (T_j)_{1 \le j \le n}$ is an n-tuple of operators the definition of the number

$$\mathcal{K}_{\mathcal{J}}(\tau) = \liminf_{A \in \mathcal{R}_1^+(\mathcal{H})} |[A, \tau]|_{\mathcal{J}}$$

from ([142] see also [141], [144]), where $(\mathcal{J}, |\ \ |_{\mathcal{J}})$ is a normed ideal and $R_{\_}^{\pm}(\mathcal{H}) = \{A \in \mathcal{R}(\mathcal{H}) \mid 0 \le A \le 1\}$ the lim inf being with respect to the natural order on $\mathcal{R}_{_}^{\pm}(\mathcal{H})$and where we use the notation$[A, \tau] = ([A, T_j])_{1 \le j \le n}$ and $\left|(X_j)_{1 \le j \le n}\right|_{\mathcal{J}} = max_{1 \le j \le n} |X_j|_{\mathcal{J}}$. If $|\ \ |_{\mathcal{J}} = |\ \ |_\Phi$ we also write $k_\Phi(\tau)$ for $k_{\mathcal{J}}(\tau)$.

We will be mainly interested in the condition $k_{\mathcal{J}}(\tau) = 0$. Results concerning this are summarized in [144]. If τ is an n- tuple of commuting hermitian operators and $\mathcal{J} = \mathcal{C}_n$ the

Schattenvon Neumann class, then we have $k_{\mathcal{C}_n}(\tau) = 0$ if $n \geq 2$. This implies the fact that $k_{\mathcal{C}_2}(N) = 0$ if N is a normal operator which underlies the results in [140].

We should also recall (see [142] or [141]) that $k_{\mathcal{J}}(\tau) = 0$ is equivalent to $k_{\mathcal{J}}(\tau \coprod \tau *^*) = 0$ where $\tau^* = (T_j^*)_{1\leq j \leq n}$ or to $k_{\mathcal{J}}(Re\,\tau \coprod 1m\tau) = 0$ where $Re\,\tau = (Re\,T_j)_{1\leq j \leq n}$ and $\mathrm{Im}\tau = (\mathrm{Im}\,T_j)_{1\leq j \leq n}$.

The condition $k_{\mathcal{J}}(\tau) = 0$ is also equivalent to the existence of a sequence $\in A_n \in \mathcal{R}_1^+(\mathcal{H})$ such that $A_n \overset{\omega}{\to} 1$ and $\left|[A_n, \tau]\right|_{\mathcal{J}} \to 0$ as $n \to \infty$ or also to the existence of a sequence $A_n \uparrow 1$, $A_n \in \mathcal{R}_1^+(\mathcal{H})$ satisfying additional conditions like $m > n \Rightarrow A_m A_n = A_n$ and $A_n B_n = B_n$ where $\mathcal{B}_n \in \mathcal{R}(\mathcal{H})$ are given and so that $\left|[A_n, \tau]\right|_{\mathcal{J}} \to 0$ as $n \to \infty$.

Let $\tau = (T_j)_{1\leq j\leq n}, T_j = T_j^*, 1 \leq j \leq n$ be an n-tuple of hermitian operators in $\mathcal{B}(\mathcal{H})$ and let $(\mathcal{J}, |\,|_{\mathcal{J}})$ be a normed ideal we define $\mathcal{E}(\tau\,;\mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, T_j] \in \mathcal{J}, 1 \leq j \leq n\}$ and $\mathcal{K}(\tau\,;\mathcal{J}) = \mathcal{E}(\tau\,;\mathcal{J}) \cap \mathcal{K}(\mathcal{H})$. Then $\mathcal{E}(\tau;\mathcal{J})$ is a Banach algebra with the norm $|||X||| = \|X\| + |[X,\tau]|_{\mathcal{J}}$ with an isometric involution $|||X^*||| = |||X|||$ and $\mathcal{K}(\tau\,;\mathcal{J})$ is a closed two-sided ideal, which is also closed under the involution. We shall denote by $\mathcal{P}(\mathcal{H})$ the finite-rank hermitian projections. Clearly $\mathcal{P}(\mathcal{H}) \subset \mathcal{R}(\mathcal{H}) \subset \mathcal{K}(\tau\,;\mathcal{J})$.

Proposition(2.3.1)[132]: Assume $\mathcal{K}_{\mathcal{J}}(\tau) = 0$.

(a) If $P \in \mathcal{P}(\mathcal{H})$ and $\epsilon > 0$, then there is $A \in R_1^+(\mathcal{H})$ so that $P \leq A$ and $|||A||| < 1 + \epsilon$.

(b) If $\mathcal{R}(\mathcal{H})$ is dense in $\mathcal{J}$ and $P \in \mathcal{P}(\mathcal{H}), \mathcal{K}_r \in \mathcal{K}(\tau\,;\mathcal{J}), 1 \leq r \leq m$ and $\epsilon > 0$, then there is $A \in \mathcal{R}_1^+(\mathcal{H})$ so that $P \leq A, |||(1-A)K_r||| < \epsilon, 1 \leq r \leq m$ and $|||A||| < 1 + \epsilon$.

Proof: (a) Since $\mathcal{K}_{\mathcal{J}}(\tau) = 0$ there is $A \in \mathcal{R}_1^+(\mathcal{H}), P \leq A$ so that $|[A,\tau]|_{\mathcal{J}} < \epsilon$, which in view of the fact that $\|A\| \leq 1$ gives $|||A||| < 1 + \epsilon$.

b) Since $[\mathcal{K}_r, T_j] \in \mathcal{J}, 1 \leq r \leq m, 1 \leq j \leq n$ and $\mathcal{R}(\mathcal{H})$ is dense in $\mathcal{J}$, there is a projection $Q \in \mathcal{P}(\mathcal{H})$ so that $|(I - Q)[K_r, T_j]|_{\mathcal{J}} < \epsilon/4$ and $\|(1 - Q)\mathcal{K}_r\| < \epsilon/4, 1 \leq r \leq m, 1 \leq j \leq n$. Clearly, we may assumewithout loss of generality that $P \geq Q$ and $|||K_r||| \leq 1, 1 \leq r \leq m$.

Using a), there is $A \in R_1^+(\mathcal{H})$ so that $Q \leq P \leq A$ and $|[A,\tau]|_{\mathcal{J}} < \epsilon/4$. We have

$$\|(I-A)K_r\| \leq \|(I-Q)K_r\| < \epsilon/4$$

and

$$|[(I - A)\mathcal{K}_r, T]|_{\mathcal{J}} \leq |[A,t]|_{\mathcal{J}}\|\mathcal{K}_r\| + \max_{1\leq j\leq n}\left|(I - A)[\mathcal{K}_r, T_j]\right|_{\mathcal{J}} < \epsilon/4 + \epsilon/4 = \epsilon/2.$$

It follows that $|||(I-A)K_r||| < \epsilon$.

Corollary(2.3.2)[132]: If $\mathcal{K}_{\mathcal{J}}(\tau) = 0$ and $\mathcal{R}(\mathcal{H})$ is dense in $\mathcal{J}$, then $\mathcal{R}(\mathcal{H})$ is dense in $\mathcal{K}(\tau;\mathcal{J})$.

Proposition(2.3.3)[132]: Assume $k_{\mathcal{J}}(\tau) = 0$ and $\mathcal{J}^{(0)} = \mathcal{J}$, that is $\mathcal{R}(\mathcal{H})$ is dense in $\mathcal{J}$. *Let* $X_1, \ldots, X_m \in \mathcal{E}(\tau\,;\mathcal{J}), \mathcal{K}_1, \ldots, \mathcal{K}_3 \in \mathcal{K}(\tau\,;\mathcal{J}), P \in \mathcal{P}(\mathcal{H})$ and $\epsilon > 0$ be given. Then there is $B \in \mathcal{R}_1^+(\mathcal{H})$ so that $P \leq B, |||B||| < 1 + \epsilon$

$$|||(I - B)K_j||| < \epsilon, |||[X_p, B]||| < \epsilon$$

for $1 \leq j \leq r, 1 \leq p \leq m$.

Proof: Without loss of generality we will assume that $X_p = X_p^*, 1 \leq p \leq m$. Since $\mathcal{J} = \mathcal{J}^{(0)}$ there is $P_0 \in \mathcal{P}(\mathcal{H})$ so that $P \leq P_0$ and

$$|(I - P_0)[X_p,\tau]|_{\mathcal{J}} + |[X_p,\tau](I - P_0)|_{\mathcal{J}} < \epsilon/2.$$

Applying repeatedly Proposition(2.3.1) we can find$P_s \in \mathcal{P}(\mathcal{H}), A_s \in \mathcal{R}_1^+(\mathcal{H})$,

$$P_0 \leq P_1 \leq P_2 \leq \dots,$$
$$A_0 \leq A_1 \leq A_2 \leq \dots$$

so that $P_s \uparrow I$ as $s \to \infty$ and $P_s \leq A_s \leq P_{s+1}, (I - P_{s+1})X_p A_s = 0$,

$$(I - P_{s+1})T_l A_s = 0, (I - P_{s+1})X_p A_s = 0$$

(that is $P_{s+1}\mathcal{H} \supset X_p A_s \mathcal{H} + T_l A_s \mathcal{H}), |||A_s||| < 1 + \epsilon 2^{-s-1}$ and $|||(I - A_s)K_j||| < \epsilon$. For $1 \leq p \leq m, 1 \leq l \leq n, 1 \leq j \leq r$and all $s \geq 0$. Let $B = N^{-1}(A_1 + \dots + A_N)$.

We will show that choosing N su□ciently large, B will have all the desired properties.

Clearly, since $A_s \geq P, 1 \leq s \leq N$ we will also have the same inequality for their mean, that is $B \geq P$. Similarly, $(I - B)K_j$ is the mean of the $(1 - A_s)K_j, 1 \leq s \leq N$ and this gives $|(\|I - B)K_j\| | < \epsilon$.

Also, the same kind of argument gives $|||B||| < 1 + N^{-1}\epsilon$.

To prove that $\||[X_p, B]|\| < \epsilon$ if N is large enough we will show that $[X_p, B] \to 0$ and $|[X_p, B], \tau]|_{\mathcal{J}} \to 0$ as $N \to \infty$. Remark that the conditions on P_s, A_s, X_p, T_l imply that in the orthogonal sum decomposition

$$\mathcal{H} = P_0\mathcal{H} \oplus (P_1 - P_0)\mathcal{H} \oplus (P_2 - P_1)\mathcal{H} \oplus \dots$$

we have that A_s is block-diagonal, while the X_p and T_1, being hermitian, are block-tridiagonal. With the notation $Q_0 = P_0, Q_s = P_s - P_{s-1}, s \geq 1$, we have $A_{s-1} = Q_0 + \dots + Q_{s-1} + Q_s A_{s-1} Q_s$ if $s \geq 1$. It follows that

$$\left\| B - \left(Q_0 + \sum_{1 \leq s \leq N} \left(1 - \frac{s-1}{N}\right) Q_s \right) \right\| = \left\| N^{-1} \sum_{1 \leq s \leq N} Q_{s+1} Q_s Q_{s+1} \right\| \leq N^{-1}.$$

Hence the tridiagonality gives

$$\|[B, X_p]\| \leq 2N^{-1} \|X_p\| + \left\|\left[Q_0 + \sum_{1 \leq s \leq N} \left(1 - \frac{s-1}{N}\right) Q_s, X_p \right]\right\|$$
$$\leq 2N^{-1} + N^{-1} \left(\left\| \sum_{1 \leq s \leq N} Q_{s+1} X_p Q_s \right\| + \left\| \sum_{1 \leq s \leq N} Q_s X_p Q_{s+1} \right\| \right) \leq 4N^{-1} \|X_p\|$$

and hence $[B, X_p] \to 0$ as $N \to \infty$.

Since we may choose $P_0 \neq 0$, we have $\|B\| = 1$ and hence $|||B||| < 1 + N^{-1}\epsilon$ gives $|[B, \tau]|_{\mathcal{J}} < \epsilon N^{-1}$ It follows that

$$|[[B, X_{\mathcal{J}}], \epsilon]|_{\mathcal{J}} \leq 2|[B, \tau]|_{\mathcal{J}} \|X_p\| + |[B, [X_p, \tau]]|_{\mathcal{J}}$$
$$\leq 2N^{-1} \epsilon \|Xp\| + |(I - B)[Xp, \tau]|I + |[Xp, \tau](I - B)|_{\mathcal{J}}.$$

Since $B \geq P_0$, it follows that

$$|(I - B)[X_p, \tau]|_p + |[X_p, \tau](I - B)|_{\mathcal{J}} < \epsilon/2.$$

Hence $|[B, Xp], \tau|_{\mathcal{J}} < \epsilon$ for N.

Corollary(2.3.4)[132]: Assum$k_{\mathcal{J}}(\tau) = 0$ and $\mathcal{J}^{(0)} = \mathcal{J}$. Let $X_1, \dots, X_m \in \mathcal{E}(\tau; \mathcal{J})$ and a sequence $Y_s \in \mathcal{R}(\mathcal{H}), s \in \mathbb{N}$ be given. Then there is a sequence $A_s \in \mathcal{R}_1^+(\mathcal{H})$ so that $A_s Y_s = Y_s$ and $A_s A_t = A_t, A_s X_p A_t = X_p A_t$ $if\ s > t$ and moreover

$$A_s \uparrow I, |||A_s||| \to 1, |||(I - A_s)K||| \to 0, |||[X_p, A_s]||| \to 0$$

as $s \to \infty$ for all $k \in \mathcal{K}(\tau; \mathcal{J})$ and $1 \leq p \leq m$.

We pass now to the quotient Banach algebra with involution $\mathcal{E}(\tau; \mathcal{J})/\mathcal{K}(\tau; \mathcal{J})$ which we shall denote by $\mathcal{E}/\mathcal{K}(\tau; \mathcal{J})$. If $p : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) = \mathcal{B}/\mathcal{K}(\mathcal{H})$ is the canonical homomorphism to the Calkin algebra, which we shall denote by $\mathcal{B}/\mathcal{K}(\mathcal{H})$, then there is a canonical isomorphism of $\mathcal{E}/\mathcal{K}(\tau\,;\,\mathcal{J})$and the sub algebra $p(\mathcal{E}(\tau\,;\,\mathcal{J}))$ of $\mathcal{B}/\mathcal{K}(\tau\,;\,\mathcal{J})$. We shall often also denote by p the homomorphism $\mathcal{E}(\tau\,;\,\mathcal{J}) \to \mathcal{E}/\mathcal{K}(\tau\,;\,\mathcal{J})$.

Proposition(2.3.5)[132]: We assume $k_{\mathcal{J}}(\tau) = 0$ and $\mathcal{J}^{(0)} = \mathcal{J}$. Given $X \in \mathcal{E}(\tau\,;\,\mathcal{J})$ and $\mathcal{E} > 0$ there is $A \in R_1^+(\mathcal{H})$ so that

$$\|A\| = 1, |||A||| < 1 + \epsilon \text{ and } |||(I - A)X||| < \|p(X)\| + \epsilon$$

where the norm of $p(X)$ is the $\mathcal{B}/\mathcal{K}(\mathcal{H})$ norm. In particular, the norm of $X = \mathcal{K}(\tau\,;\,\mathcal{J})$ in $\mathcal{E}/\mathcal{K}(\tau\,;\,\mathcal{J})$ equals the norm of $p(X)$ in $\mathcal{B}/\mathcal{K}(\mathcal{H})$. Thus algebraic embedding of $\mathcal{E}/\mathcal{K}(\tau\,;\,\mathcal{J})$ into $\mathcal{B}/\mathcal{K}(\mathcal{H})$ is isometric and $\mathcal{E}/\mathcal{K}(\tau\,;\,\mathcal{J})$ identifies with a C^*-subalgebra of $\mathcal{B}/\mathcal{K}(\mathcal{H})$.

Proof: We have stated this fact which is an immediate generalizationof results in [140] and [134], with a lot of detail, since it will be often used.

In view of our assumption, that $k_{\mathcal{J}}(\tau) = 0$, there are $A_n \uparrow I, A_n \in \mathcal{R}_1^+(\mathcal{H})$ so that $|[A_n, \tau]|_{\mathcal{J}} \to 0$ as $n \to \infty$. then also $\|(I - A_n)X\| \to \|p(X)\|$ as $n \to \infty$.we also have

$$|[(I - A_n)X, \tau]|_{\mathcal{J}} \le |[A_n, \tau]|_{\mathcal{J}} \|X\| + |(I - A_n)[X, \tau]|_{\mathcal{J}}$$

and the first term in the right-hand side $\to 0$ as $n \to \infty$ by the properties of the A_n , while the second also$\to 0$ since $\mathcal{J}^{(0)} = \mathcal{J}$ and $(I - A_n)[X, \tau]$ converges weakly to 0 as $n \to \infty$. The rest of the statement is well explained in the statement of the corollary itself.

We show what amounts to countable degree-1 saturation of $\mathcal{E}/\mathcal{K}(\tau\,;\,\mathcal{J})$ under the assumption that $k_{\mathcal{J}}(\tau) = 0$, in the model-theory terminology of [134],[114].The result is given in Theorem (2.3.8),which is formulated in operator-algebra terms, using one of the equivalent definitions of countable degree-1 saturationwhich can be found in [114].

We begin with a rather standard technical fact.

Lemma (2.3.6)[132]: Let $G = G^* \in \mathcal{E}(\tau\,;\,\mathcal{J})$be such that$|||G - \frac{3}{2} I||| \le 1$.

Then $G^{1/2} \in \mathcal{E}(\tau\,;\,\mathcal{J})$ and there is a universal constant C, so that

$$|||[G^{1/2}, X]||| \le C|||[G, X]||| \text{ if } X \in \mathcal{E}(\tau\,;\,\mathcal{J})$$

and

$$|[G^{1/2}, \tau]|_{\mathcal{J}} \le C|[G, \tau]|_{\mathcal{J}}.$$

Proof: The Lemma is an easy consequence of the functional calculus formula

$$G^{1/2} = (2\pi i)^{-1} \int_{\Gamma} (zI - G)^{-1} z^{1/2} dz,$$

where Γ is the circle $|z - 3/2| = 5/4$, and of the fact that for $z \in \Gamma$ we have :

$$|||(zI - G)^{-1}||| = 4/5|||(zI - G)^{-1}(4/5(z - 3/2)I - 4/5(G - 3/2I))^{-1}|||$$
$$\le (1 - 4/5)^{-1} = 5$$

and

$$[(zI - G)^{-1}, X] = (zI - G)^{-1}[G, X](zI - G)^{-1}$$

Lemma(2.3.7)[132]: Assume $k_{\mathcal{J}}(\tau) = 0$ and $\mathcal{J} = \mathcal{J}^{(0)}$. Let $M_n \in \mathcal{E}(\tau\,;\,\mathcal{J})$, $n \in N, \epsilon_m \downarrow 0$ as $m \to \infty, P_k \in \mathcal{P}(\mathcal{H}), P_k \uparrow I$ as $k \to \infty$ and and increasing function $\varphi : \mathbb{N} \to \mathbb{N}$be given. Then there are $R_m \in \mathcal{R}_1^+(\mathcal{H}), m \in \mathbb{N}$ so that

(i) $\sum_{m\ge1} R_m^2 = 1$

(ii) the R_m's commute

(iii) $\|R_m\| = 1$ and $|[R_m, \tau]|_{\mathcal{J}} < \epsilon_m$ if $m \ge 2$

(iv) $R_m P_n = 0$ if $m \ge n + 2, n \ge 1$

(v) $|||[R_m, M_k]|||| < \epsilon_m$ if $k \le \varphi(m), m \ge 2$
(vi) $R_n M_k R_m = 0, R_n R_m = 0$
If $k \le \varphi(m), k \le \varphi(n), |n - m| \ge 2, m \ge 2, n \ge 2$.
Proof: There will be no loss of generality to assume that $M_1 = I$ and $M_k = M_k^*, K \in \mathbb{N}$. Given $\delta_m < 1/10\, \delta_m \downarrow 0\ m \to \infty$, we can use Proposition (2.3.1) repeatedly, to find a sequence of projections $\mathrm{E_k} \in \mathcal{P}(\mathcal{H}), \mathrm{E_k} \uparrow I$ as $k \to \infty$ and a sequence of $A_k \in \mathcal{R}_1^+(\mathcal{H}), A_k \uparrow I$ as $k \to \infty, A_1 = 0$ satisfying the following conditions:

$$\mathrm{E_k} \ge P_k, \mathrm{E_k} \ge A_k, (I - \mathrm{E_k}) M_p A_k = 0 \text{ if } p \le \varphi(k + 2)$$

and also

$$\mathrm{A_{k+1}} \ge \mathrm{E_k}, \|\mathrm{A_{k+1}} - \mathrm{E_k}\| = 1,$$
$$|||A_{k+1}||| < 1 + \delta_{k+1}$$

and

$$\left|\left|\left|[A_{k+1}, M_p]\right|\right|\right| < \delta_{k+1} \text{ if } p \le \varphi(k + 2).$$

Note that since the $\mathrm{E_k}$'s are projections, $0 = \mathrm{A_1} \le \mathrm{E_1} \le \mathrm{A_2} \le \mathrm{E_2} \le$ z $\cdots \le E_k \le A_{k+1} \le E_{k+1} \le \cdots \le I$ implies that $\{\mathrm{A_k} | \mathrm{k} \ge 1\} \cup \{\mathrm{E_k} | \mathrm{k} \ge 1\}$ is a set of commuting operators.

With these preparations one would be tempted to define R_n to be $(A_n - A_{n-1})^{1/2}$,however this would lead to di□culties with commutators and $|||\cdot|||$ -norms because the square-root function is not di□erentiable at 0.

We define

$$B_n = I - (I - A_n^2)^2 = A_n^2(2I - A_n).$$

Then $E_n \le A_{n+1} \le E_{n+1}$ easily gives $E_n \le B_{n+1} \le E_{n+1}$ and $A_1 = 0$ gives $B_1 = 0$. It is also easily seen that defining

$$R_n = (B_n - B_{n-1})^{1/2}, n \ge 2$$
$$R_1 = 0$$

we have

$$B_n^{1/2} = A_n(2I - A_n^2)^{1/2}$$
$$(I - B_{n-1})^{1/2} = I - A_n^2$$

$$R_n = (B_n(I - B_{n-1}))^{1/2} = B_n^{1/2}(I - B_{n-1})^{1/2}$$
$$= A_n(2I - A_n^2)^{1/2}(I - A_{n-1}^2).$$

Then for $n \ge 2$ we have

$$|||(2I - A_n^2) - 3/2I\,||| = |||A_n^2 - 1/2I\,|||$$
$$= \|A^2 - 1/2I\| + |[A_n^2, \tau\,]|_{\mathcal{J}}$$
$$\le 1/2 + 2\,\|A_n\|\,|[A_n, \tau\,]|_{\mathcal{J}} \le 1/2 + 2\delta_n \le 1.$$

We can then apply Lemma(2.3.6) to $G = 2I - A_n^2$ and $X \in \mathcal{E}(\tau\,; \mathcal{J})$ and get that

$$|[(2I - A_n^2)^{1/2}, \tau\,]|_{\mathcal{J}} \le C|[A_n^2, \tau\,]|_{\mathcal{J}} \le 2C|[A_n, \tau\,]|I \le 2C\delta_n$$

and

$$\left|\left|\left|[(2I - A_n^2)^{1/2}, X]\right|\right|\right| \le C\,|||[A_n^2, X]\,||| \le 2C\,|||[A_n\,, X]|||(1 + \delta_n) \le$$
$$3C|||A_n\,, X\,|||\,.$$

Since $P_m \uparrow I$ we have $E_m \uparrow I$ and hence $\mathrm{B_m} \uparrow I$ as $m \to \infty$. In view of $\mathrm{B_1} = 0$ we get that condition (i) is satisfied by the $\mathrm{R_k}$. Also, since the $\mathrm{B_m}$ commute, it follows that the $\mathrm{R_m}$ commute and that condition (ii) holds. Further, since $\|A_{k+1} - E_k\| = 1$, we have that A_{k+1}

has an eigenvector for the eigenvalue (i) in $(E_{k+1} - E_k)\mathcal{H}$, which is then also an eigenvector for the eigenvalue (i) of B_{k+1} and an eigenvector for the eigenvalue 0 of B_k, so that it is an eigenvector for the eigenvalue (i) for $(B_{k+1} - B_k)^{1/2} = R_{k+1}$. Thus we have $\|R_m\| = 1$ if $m \geq 2$, which is the first part of condition (ii). Further $R_m E_{m-2} = 0$ gives $R_m R_{m-2} = 0$, so that (iv) is satisfied. Also, since $M_1 = I$ and $M_k = M_k^*$, to check that (vi) holds, it su□ces to check that $R_n M_k R_m = 0$ if $n \geq m + 2, 1 \leq k \leq \varphi(m)$.indeed, we have $R_n M_k R_m = B_n^{1/2}(I - B_{n-1})^{1/2} M_k B_m^{1/2} (I - B_{m-1})^{1/2}$ and thus it su□ces to show that $(1 - B_{n-1})M_k B_m = 0$ if $1 \leq k \leq \varphi(m), n \geq m + 2$, which in turn will follow if we show that $(I - A_{n-1})M_k A_m = 0$ if $1 \leq k \leq \varphi(m), n \geq m + 2$. Note further that if$n \geq m + 2$ we have$A_{n-1} \geq E_{n-2} \geq E_m$ and it su□ces if $(I - E_m)M_k A_m$for $k \leq \varphi(m)$ for $m \geq 2$, which is satisfied in view of the construction of the E_m and A_m.

We are thus left with having to deal with the second part of (iii) an (v).

We have

$$|[R_m \tau]|_{\mathcal{J}} = |[A_m(2I - A_m^2)^{1/2}(I - A_{m-1}^2), \tau]|_{\mathcal{J}}$$

$$\leq 2||[R_m \tau]|_{\mathcal{J}} + |[(2I - A_m^2)^{1/2}, \tau]|_{\mathcal{J}} + 2|[A_{m-1}^2, \tau]|_{\mathcal{J}}$$

$$\leq 2\delta_m + 2C\delta_m + 4\delta_{m-1}.$$

Hence, choosing the δ_m's so that $2(C + 1)\delta_m + 4\delta_{m-1} < \epsilon_m$will insure that the second part of (iii) holds.

Turning to condition (vi), we have

$$\begin{aligned}
|||[R_m, M_k]||| &= \left|\left|\left|[A_m(2I - A_m^2)^{1/2}(I - A_{m-1}^2), M_k]\right|\right|\right| \\
&\leq |||[A_m, M_k]||| \left|\left|\left|(2I - A_m^2)^{1/2}\right|\right|\right| \, |||I - A_{m-1}^2||| \\
&+ \left|\left|\left|[(2I - A_m^2)^{1/2}, M_k]\right|\right|\right| \, |||A_m||| \, |||I - A_{m-1}^2||| \\
&+ 2 \, |||[A_{m-1}, M_k]||| \, |||A_{m-1}||| \, |||A_m||| \left|\left|\left|(2I - A_m^2)^{1/2}\right|\right|\right| \\
&\leq \delta_m(2 + |[(2I - A_m^2)^{1/2}, \tau]|_{\mathcal{J}})(2 + \delta_{m-1})^2 + 3C \, |||[A_m, M_k]||| \, (1 \\
&+ \delta_m)(2 + \delta_{m-1})^2 + 2\delta_{m-1}(1 + \delta_{m-1})(1 + \delta_m)(2 \\
&+ |[(2I - A_m^2)^{1/2}, \tau]|_{\mathcal{J}}) \\
&\leq \delta m(2 + 2C\delta_m)(2 + \delta_{m-1})^2 + 3C\delta m(1 + \delta_m)(2 + \delta_{m-1})^2 \\
&+ 2\delta_{m-1}(1 + \delta_{m-1})(1 + \delta_m)(2 + 2C\delta_m),
\end{aligned}$$

if $m \geq 2$ and $k \leq \varphi(m)$. Clearly, the δ_m 's can be chosen so that $|||[R_m, M_k]||| < \epsilon_m$.

By Proposition (2.3.5), $\mathcal{E}/\mathcal{K}(T ; \mathcal{J})$ under the assumptions that $k_{\mathcal{J}}(T) = 0$ and $\mathcal{J} = \mathcal{J}^{(0)}$ is a C^*-algebra, actually a C^*-subalgebra of the Calkin al-gebra. Recall also that p will denote both the homomorphis $\mathcal{B}(\mathcal{H}) \to \mathcal{B}/\mathcal{K}(\mathcal{H})$ as well as the homomorphism $\mathcal{E}(T ; \mathcal{J}) \to \mathcal{E}/\mathcal{K}(T ; \mathcal{J}$, which can be viewed as its restriction to $\mathcal{E}(T ; \mathcal{J})$ (see the discussion preceding Proposition (2.3.3) and Proposition (2.3.5).

Let $X_j, X_j^*, j \in \mathbb{N}$ be non-commuting indeterminates and let

$$f_n(X_1, \dots, X_n) = e_n + \sum_{j=1}^{n} \left(a_{jn} X_j b_{jn} + c_{jn} X_j^* d_{jn}\right)$$

where $e_n, a_{1n}, \dots, a_{nn}, \dots, b_{1n}, \dots, b_{nn}, c_{1n}, \dots, c_{nn}, d_{1n}, \dots, d_{nn}$ are in $\mathcal{E}/\mathcal{K}(t ; \mathcal{J})$ so that the f_n are non-commutative polynomials with coefficients not commuting with the variables. We shall denote the ring of such polynomials by $\mathcal{E}/\mathcal{K}(t; \mathcal{J}) \langle X_j X_j^* | j \in \mathbb{N} \rangle$, , the f_n's being polynomials of degree ≤ 1 in the indeterminates.

Theorem (2.3.8)[132]: Assume $k_{\mathcal{J}}(t) = 0$ and $\mathcal{J} = \mathcal{J}^{(0)}$. Let

$$e_n, a_{jn}, b_{jn}, c_{jn}, d_{jn} \in \frac{\mathcal{E}}{\mathcal{K}(t,\mathcal{J})}, 1 \leq j \leq n, n \in \mathbb{N},$$

be such that there are $\mathcal{Y}_{jn} \in \mathcal{E}/\mathcal{K}(t;\mathcal{J})$,$1 \leq j \leq n, n \in \mathbb{N}$, so that $\|\mathcal{Y}_{jn}\| <$ 1 and $\left|\left\|en + \sum_{1\leq j\leq n}\left(a_{jn}\mathcal{Y}_{jm}b_{jn} + c_{jn}\mathcal{Y}_{jm}^{*}d_{jn}\right)\right\| - r_n\right| < 1/m$, if $1 \leq n \leq m$ where $r_n \in \mathbb{R}$. then there are $\mathcal{Y}_i \in \mathcal{E}/\mathcal{K}(t;\mathcal{J})$, $j \in \mathbb{N}$ so That $\|\mathcal{Y}_i\| \leq 1$, for $j \in \mathbb{N}$ and

$$\left\|e_n + \sum_{1\leq j\leq n}\left(a_{jn}\mathcal{Y}_j b_{jn} + c_{jn}\mathcal{Y}_j^{*}d_{jn}\right)\right\| = r_n$$

for all $n \in \mathbb{N}$.

Proof: Let $E_n, A_{jn}, B_{jn}, C_{jn}, D_{jn}, Y_{jn} \in \mathcal{E}(\tau;\mathcal{J})$ for $1 \leq j \leq n, n\, \mathbb{N}$, be so that

$$p(E_n) = e_n, p(A_{jn}) = a_{jn}, p(B_{jn}) = b_{jn}, p(C_{jn}) = c_{jn}, p(D_{jn}) = d_{jn}, p(Y_{jn}) = \mathcal{Y}_{jn}, \left|\left\|Y_{jn}\right\|\right| < 1$$

and $|[Y_{jn},\tau]|_{\mathcal{J}} < \epsilon_n$ for some given sequence $\epsilon_n \downarrow 0$. It will be convenient to also introduce $f_n(X_1,\ldots,X_n) \in \mathcal{E}/\mathcal{K}(\tau;\mathcal{J})\langle X_j, X_j^{*} \mid j \in \mathbb{N}\rangle$ and $F_n(X_1,\ldots,X_n) \in \mathcal{E}(\tau;\mathcal{J})\langle X_j, X_j^{*} \mid j \in \mathbb{N}\rangle$ the non-commutative polynomials

$$f_n(X_1,\ldots,X_n) = e_n + \sum_{1\leq j\leq n}\left(a_{jn}X_j b_{jn} + c_{jn}X_j^{*}d_{jn}\right),$$

$$F_n(X_1,\ldots,X_n) = E_n + \sum_{1\leq j\leq n}\left(A_{jn}X_j B_{jn} + C_{jn}X_j^{*}D_{jn}\right).$$

We shall apply Lemma (2.3.7) with a sequence $M_k \in E(\tau;\mathcal{J}), k \in \mathbb{N}$ and an increasing function $\varphi: \mathbb{N} \to \mathbb{N}$ such that the set $\{M_k \mid 1 \leq k \leq \varphi(m)\}$ cntains the following operatr $E_m, A_{jm}, B_{jm}, C_{jm}, D_{jm}, Y_{jm}, Y_{jm}^{*}$ where $1 \leq j \leq m$ and also $F_n(Y_{1m},\ldots,Y_{nm})$ and

$$(F_p(Y_{1m},\ldots,Y_{pm}))^{*}F_q(Y_{1m},\ldots,Y_{qm})$$

with $1 \leq n \leq m, 1 \leq p \leq m, 1 \leq q \leq m$. Note that the listed operators won't exhaust $\{M_k \mid 1 \leq k \leq \varphi(m)\}$, since φ being increasing we will have that if $1 \leq m' < m$ then $\{M_k \mid 1 \leq k \leq \varphi(m')\} \subset \{M_k \mid 1 \leq k \leq \varphi(m)\}$.

Since $p(F_n(Y_{1m},\ldots,Y_{nm})) = f_n(\mathcal{Y}_{1m},\ldots,\mathcal{Y}_{nm})$ if $1 \leq n \leq m$, we can find $P_k \in \mathcal{P}(\mathcal{H}), P_k \uparrow I$ so that

$$|\|F_n(Y_{1m},\ldots,Y_{nm})(I - P_m)\| - r_n| < 1/m$$

if $1 \leq n \leq m$. Remark that if $1 \leq n \leq m$ and $N \geq m$ then

$$\left|\left\|F_n(Y_{1m},\ldots,Y_{nm})\sum_{k\geq N+2}R_k^2\right\| - r_n\right| < 1/m$$

because $\sum_{k\geq N+2}R_k^2 \leq I - P_m$ and $I - \sum_{k\geq N+2}R_k^2 \in \mathcal{R}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ so that

$$\|F_n(Y_{1m},\ldots,Y_{nm})(I - P_m)\| \geq \left\|F_n(Y_{1m},\ldots,Y_{nm})\sum_{k\geq N+2}R_k^2\right\| \geq \|f_n(\mathcal{Y}_{1m},\ldots,\mathcal{Y}_{nm})\|$$

We can therefore find a sequence $1 < N_1 < N_2 <\ldots$ so that $N_m \geq m + 2, N_{p+1} - N_p \geq 8$ for all $m, p \in \mathbb{N}$ and

$$\left| \left\| F_n(Y_{1m},\dots,Y_{nm}) \sum_{N_m \le k < N_{m+1}} R_k^2 \right\| - r_n \right| < 1/m$$

if $1 \le n \le m$ and also

$$\left| \left\| F_n(Y_{1m},\dots,Y_{nm}) \sum_{N_m+3 \le k < N_{m+1}-3} R_k^2 \right\| - r_n \right| < 1/m.$$

We will show that if the ϵ_m are chosen so that $\sum_{m\ge1} \epsilon_m < \infty$, then the operators

$$Y_j \sum_{m \ge j} \left(\sum_{N_m \le k < N_{m+1}} R_k Y_{jm} R_k \right)$$

will satisfy $Y_j \in \mathcal{E}(T\,;\mathcal{I}), \|Y_j\| \le 1$ and $p(Y_j) = y_j$ will satisfy $f_n(y_1,\dots,y_n) = r_n$ for all $n \in \mathbb{N}$.

We will not need to put conditions on the ϵ_m in order that $\|Y_j\| \le 1$.

Indeed, this can be seen as follows. Let $Z : \mathcal{H} \to \mathcal{H} \otimes l^2(N)$ be the operator

$$Zh = \sum_{m \ge j} \sum_{N_m \le k < N_{m+1}} R_k h \otimes e_k$$

And let $S_j \in \mathcal{B}(\mathcal{H} \otimes l^2(\mathbb{N}))$ be the operator

$$S_j \sum_{k \ge 1} h_k \otimes e_k = \sum_{m \ge j} \sum_{N_m \le k < N_{m+1}} Y_{jm} h_k \otimes e_k.$$

Since $\|Y_{jm}\| < 1$, we have $\|S_j\| \le 1$ and we also have $\|Z\| \le 1$ since

$$Z^*Z = \sum_{m \ge j} \sum_{N_m \le k < N_{m+1}} R_k^2 \le \sum_{k \ge 1} R_k^2 = I.$$

Hence $\|Y_j\| \le 1$ since $Y_j = Z^* S_j Z$.

Our next task will be to show that if $\sum_{m\ge1} \epsilon_m < \infty$, we will have $|[Y_j, \tau\,]|_{\mathcal{J}} < \infty$, which together with the boundedness of Y_j we just showed, will give $Y_j \in \mathcal{E}(\tau\,;\mathcal{I})$.

Since the sum defining Y_j is weakly convergent to Y_j,it will be sufficient to show that assuming $\sum_{m\ge1} \epsilon_m < \infty$, we can insure that

$$\sum_{m \ge j} \left\| \left[\sum_{N_m \le k < N_{m+1}} R_k Y_{jm} R_k, \tau \right] \right\|_{\mathcal{J}} < \infty.$$

Since the Y_{jm} with $1 \le j \le m$ are among the M_p with $1 \le p \le \varphi(m)$ we infer from condition (vi) in Lemma (2.3.7) that $|||[R_k, Y_{jm}]||| < \epsilon_k$if $N_m \le k$ and $1 \le$ j $\le$ m.

Also by condition (iii) of Lemma(2.3.7), $|\,\|R_k\,\||| < 1 + \epsilon_k$.

This gives

$$\left\| \left[\sum_{N_m \le k < N_{m+1}} R_k Y_{jm} R_k, \tau \right] \right\|_{\mathcal{J}} - \left\| \left[\sum_{N_m \le k < N_{m+1}} Y_{jm} R_k^2, \tau \right] \right\|_{\mathcal{J}} \le \left\| \left[\sum_{N_m \le k < N_{m+1}} [R_k, Y_{jm}] R_k \right] \right\|_{\mathcal{J}}$$

$$\le \sum_{N_m \le k < N_{m+1}} |||[R_k, Y_{jm}]||| \, |||R_k||| \le \sum_{N_m \le k < N_{m+1}} \epsilon_k (1 + \epsilon_k)$$

Hence in order that $Y_j \in \mathcal{E}(\tau\,;\,\mathcal{J})$ it will su□ce that $\sum_{k\geq 1} \epsilon_k < \infty$ and

$$\sum_{m\geq j} \left\| \left[\sum_{N_m \leq k < N_{m+1}} Y_{jm} R_k^2, \tau \right] \right\|_{\mathcal{J}} < \infty.$$

Since $|[R_k \;, \tau]|_{\mathcal{J}} < \epsilon_k$ if $k \geq 2$ by (iii) of Lemma (2.3.7) and $\|Y_{jm}\| < 1$, we have

$$\sum_{m\geq j} \left\| \left[\sum_{N_m \leq k < N_{m+1}} Y_{jm} R_k^2, T \right] \right\|_{\mathcal{J}} \leq \sum_{m\geq j} \left|[Y_{jm} \,, T]\right|_{\mathcal{J}} \cdot \left\| \sum_{N_m \leq k < N_{m+1}} R_k^2 \right\| = \sum_{k\leq 2} \left|[R_k^2, T]\right|_{\mathcal{J}}$$

$$\leq \sum_{m\geq j} \epsilon_m + \sum_{k\geq 2} 2\epsilon_k < \infty$$

under the assumption that $\sum_{k\geq 1} \epsilon_k < \infty$. Hence under this condition on the ϵ_m we have $Y_j \in \mathcal{E}(\tau\,;\mathcal{J})$.

Finally, we turn to showing that assuming $\sum_{m\geq 1} \epsilon_m < \infty$, we will have $\|f_n(y_1, \dots, y_n)\| = r_n$ for all $n \in \mathbb{N}$, where $y_j = p(Y_j)$. Clearly $f_n(y_1, \dots, y_n) = p(F_n(Y_1, \dots, Y_n))$. Note also that the relations we're aiming at being about norms in the Calkin algebra, we will no longer have to deal with $|\|.\||$-norms and the ideal $\mathcal{J}$ for this matter.

We begin by showing that we can arrange that the di□erence between $F_n(Y_1, \dots, Y_n)$ and

$$\sum_{m\geq n} \sum_{N_m \leq k < N_{m+1}} R_k F_n(Y_{1m}, \dots\dots, Y_{nm}) R_k$$

is a compact operator. Since

$$F_n(Y_1, \dots\dots, Y_n) = E_n + \sum_{1\leq j\leq n} \left(A_{jn} Y_j\, B_{jn} + C_{jn} Y_j^* D_{jn}\right)$$

it will su□ce to prove the assertion in (iii) cases, when $F_n(Y_1, \dots, Y_n)$ equals $E_n, A_{jn} Y_j B_{jn}, C_{jn} Y_{jn}^* D_{jn}$ where $1 \leq j \leq n$.In the first case we have

$$\sum_{k\geq N_n} R_k E_n R_k - E_n = - \sum_{1\leq j\leq n} \sum_{k\geq 1} R_k E_n R_k + \sum_{1\leq k<N_n} (R_k E_n R_k - E_n R_k^2).$$

The first sum being finite rank, we need that the second sum be compact.

If $k \geq n, \|[R_k, E_n]\| < \epsilon_k$ since E_n is among the M_{p} with $p \leq \varphi(n) \leq \varphi(k)$ and condition 5^0 of Lemma (2.3.7) holds. Thus, $\|R_k E_n R_k - E_n R_k^2\| < \epsilon_k$ implies that the di□erence we consider will be compact if $\sum_{k\geq 1} \epsilon_k < \infty$.

In case F_n is $A_{jn} Y_j B_{jn}$, where $1 \leq j \leq n$, we must insure compa ness of

$$\sum_{m\geq n} \sum_{N_m \leq k < N_{m+1}} R_k A_{jn} Y_{jm} B_{jn} R_k - \sum_{m\geq n} A_{jn} \left(\sum_{N_m \leq k < N_{m+1}} R_k Y_{jm} R_k \right) B_{jn}$$

$$= \sum_{m\geq n} \sum_{N_m \leq k < N_{m+1}} \left([R_k A_{jn}] Y_{jm} B_{jn} R_k + A_{jn} R_k Y_{jm} [B_{jn}, R_k] \right).$$

The last sum being a sum of finite rank operators it will su□ce to have Convergence of the sum of their norms. Since the A_{jn} and B_{jn} are among the M_{p} with $\mathrm{p} \leq \varphi(\mathrm{n}) \leq \varphi(\mathrm{k})$ we have that the norms of the commutators are majorized By ϵ_k in view of (vi) in Lemma (2.3.7) and hence the sum of norms is majorized by

$$K\sum_{K\geq 1}\epsilon_k$$

where K is a bound for $\|A_{jn}\|$ and $\|B_{jn}\|$. Thus again it will su□ce that $\sum_{k\geq 1}\epsilon_k < \infty$..

The third situation when we consider $C_{jn}Y_j^{*}D_{jm}$ is entirely analogous to that of $A_{jn}Y_jB_j$, since we treated $Y_{jn}$and Y_{jn}^{*} symmetrically in our assumptions about φ.

Again, summability of the ϵ_m will su□ce.

We need then to show that if $\sum_{k\geq 1}\epsilon_k < \infty$, we will also have that the essential norm of

$$\Omega_n = \sum_{m\geq n}\sum_{N_m\leq k<N_{m+1}} R_kF_n(Y_{1m},\dots,Y_{nm})R_k$$

will be r_n.

Using again the operator

$$Z:\mathcal{H}\to \mathcal{H}\otimes l^2(N), Zh=\sum_{k\geq N_n} R_kh\otimes e_k$$

we have $\|Z\|\leq 1$ and $Z^{*}\Gamma_{nt}Z-\Omega_n\in\mathcal{R}(\mathcal{H})$, where for $t\geq n$ we define on $\mathcal{H}\otimes l^2(\mathbb{N})$ operators Γ_{nt} by

$$\Gamma_{nt}\sum_{k\geq 1}h_k\otimes e_k=\sum_{m\geq n}\sum_{N_m\leq k<N_{m+1}}F_n(Y_{1m},\dots,Y_n)(I-P_m)h_k\otimes e_k.$$

Since $\|\Gamma_{nt}\| = sup_{m\geq t}\|F_n(Y_{1m},\dots,Y_{nm})(I-P_m)\|$ we have $|\,\|\Gamma_{nt}\|-r_n| < t^{-1}$ and hence $\lim_{t\to+\infty}\|\Gamma_{nt}\| = r_n$.This gives $\|p(\Omega_n)\|\leq \lim_{t\to+\infty}\|\Gamma_{nt}\| = r_n$ and hence we are left with the opposite inequality $\|p(\Omega_n)\|\leq r_n$.

We will again use a compact perturbation and pass from $\Omega_n$to another operator

$$\Xi_n = \sum_{m\geq n}F_n(Y_{1m},\dots,Y_{nm})\sum_{N_m\leq k<N_{m+1}}R_k^2.$$

Indeed we have

$$\Xi_n-\Omega_n=\sum_{m\geq n}\sum_{N_m\leq k<N_{m+1}}[F_n(Y_{1m},\dots,Y_{nm}),R_k]R_k.$$

And

$$[R_n(Y_{1m},\dots,Y_{nm}),R_k]R_k < \epsilon_k.$$

Again compactness will follow if $\sum_{k\geq 1}\epsilon_k<\infty$.

Recall now that we had chosen the N_m so that

$$\left|\left\|F_n(Y_{1m},\dots,Y_{nm})\sum_{N_m\leq k<N_{m+1}}R_k^2\right\|-r_n\right|<1/m$$

and also

$$\left|\left\|F_n(Y_{1m},\dots,Y_{nm})\sum_{N_{m+3}\leq k<N_{m+1-3}}R_k^2\right\|-r_n\right|<1/m.$$

Since by Lemma (2.3.7) we have that the R_k are finite rank positive contractions, commute and satisfy $|k-l|\geq 2\Rightarrow R_kR_l=0$ it is easily seen that if Δ_m is the projection onto the range of $\sum_{N_{m+3\leq k<N_{m+1-3}}}R_k^2$ we will have $R_s\Delta_m=0$ if $s<N_m$ or $s\geq N_{m+1}$ and hence

$\Delta_p\Delta_q = 0$ if $p \neq q$ and if $n \leq m$ we have

$$\left\|F_n(\mathrm{Y}_{1\mathrm{m}},\ldots,\mathrm{Y}_{\mathrm{nm}})\sum_{\mathrm{N}_{\mathrm{m}+3}\leq \mathrm{k}<\mathrm{N}_{\mathrm{m}+1}-3}\mathrm{R}_{\mathrm{k}}^2\right\| \leq \|\Xi_n\Delta_m]\| \leq \left\|F_n(\mathrm{Y}_{1\mathrm{m}},\ldots,\mathrm{Y}_{\mathrm{nm}})\sum_{\mathrm{N}_{\mathrm{m}}\leq \mathrm{k}<\mathrm{N}_{\mathrm{m}+1}}\mathrm{R}_{\mathrm{k}}^2\right\|$$

so that

$$|\|\Xi_{\mathrm{n}}\Delta_{\mathrm{m}}\| - \mathrm{r}_{\mathrm{n}}| < 1/m.$$

This implies

$$\mathrm{p}(\Xi_{\mathrm{n}}) \geq \lim_{\mathrm{m}\to+\infty}\mathrm{Sup}\|\Xi_{\mathrm{n}}\Delta_{\mathrm{m}}\| = \mathrm{r}_{\mathrm{n}}\,.$$

We will sometimes also deal with normed ideals in which the finite rank operators are not dense, which occurs when the norming function Φ is not mononorming (see the preliminaries and [9]or [139]). We begin with a basic lemma.

Lemma (2.3.9)[132]: Let Φ be a norming function and let $(\mathcal{J}_,|\,|_{\mathcal{J}}) = (\wp_{\Phi}, |\,|_{\Phi})$.so that $(\mathcal{J}^{(0)}{}_,|\,|_{\mathcal{J}}) = (\wp_{\Phi}^{(0)}, |\,|_{\Phi})$ is the closure of $\mathcal{R},(\mathcal{H})$ in $\mathcal{J}$.Assume $\mathcal{K}_{\mathcal{J}}(t) = 0$.Then $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$ is a closed two-sided ideal in $\mathcal{E}(\tau\,;\,\mathcal{J})$ and the norm in $\mathcal{E}(\tau\,;\,\mathcal{J})$ extends the norm in $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$. Moreover, the unit ball of $(\mathcal{E}(\tau\,;\mathcal{J}), |\|.\||)$ is weakly compact.

Proof. It is clear that the norm of $\mathcal{E}(\tau\,;\,\mathcal{J})$extends the norm of $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$ and that $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$is a closed subalgebra of$\mathcal{E}(\tau\,;\,\mathcal{J})$. By Corollary(2.3.2) $\mathcal{R},(\mathcal{H})$is dense in $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$ and hence $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$is the closure in $\mathcal{E}(\tau\,;\,\mathcal{J})$ of the two-sided ideal $\mathcal{R},(\mathcal{H})$, which implies that also $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)}$ is a two-sided ideal in $\mathcal{E}(\tau\,;\,\mathcal{J})$. If X is the weak limit of the net $(X_\alpha)_{\alpha\in 1}$ in the unit ball of $\mathcal{E}(\tau\,;\,\mathcal{J})$then by the weak compactness of the unit balls of $\mathcal{B}(\mathcal{H})$ and of $\wp_{\Phi}$ (see [137]) we have $\|X\| \leq 1$ and $|[X,\tau\,]|_{\Phi} \leq 1$ so that $X \in \mathcal{E}(\tau\,;\,\mathcal{J})$. Since $\mathcal{H}$ is separable we may replace $(\mathrm{X}_\alpha)_{\alpha\in I}$ by a subsequence and use the semicontinuity properties of $\|\quad\|$and $|\,|_{\Phi}$ under weak convergence to get that $|\|X\|| \leq 1$. Thus the unit ball of $\mathcal{E}(\tau\,;\,\mathcal{J})$ is a closed subset of the unit ball of $\mathcal{B}(\mathcal{H})$ and hence weakly compact.

We pass now to bounded multipliers $\mathcal{M}(\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)}))$ that is double centralizer pairs $(T'\,,T''\,)$ of bounded linear maps $\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)}) \to \mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$ so that $T'(x)y = xT''(y)$ ([138]).

Proposition (2.3.10)[132]: Assume $k_{\mathcal{J}}(\tau) = 0$ where $\mathcal{J} = \wp_{\Phi}$ and $\mathcal{J}^{(0)} = \wp_{\Phi}^{(0)}$We have $\mathcal{M}(\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})) = \mathcal{E}(\tau\,;\mathcal{J})$, that is if $(T'\,,T''\,) \in \mathcal{M}(\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)}))$ then there is a unique $T \in \mathcal{E}(\tau\,;\,\mathcal{J})$ so that $T'\,(x) = xT$ and $T''(x) = T\,x$.

Proof: By Corollary (2.3.4) there is a sequence $A_s \in \mathcal{R}_1^+(\mathcal{H})$ so that $\|A_s\| = 1, s > t \Rightarrow A_sA_t = A_t$ and $A_s \uparrow I, |\|A_s\|| \to 1, |\|(I - A_s)K\|| \to 0$ if $s \to \infty$ and $K \in \mathcal{K}(\tau\,;\,I^{(0)})$.

Assume $(T'\,,T''\,) \in \mathcal{M}(\mathcal{K}(\tau\,;\,\mathcal{J}^{(0)}))$ and let $K_s = T'(A_s)A_s = A_sT''(A_s)$.

Clearly $sup_{s\in\mathbb{N}}\,|\|K_s\|| < \infty$ the multiplier being bounded. Remark also that $s > t \Rightarrow A_tK_sA_t = A_tT'(A_s)A_sA_t = A_tT'(A_s)A_t = A_tA_sT''(A_t) = A_tT''(A_t) = K_t$. Hence if T is the weak limit of a subsequence of the K_s, we have $A_tT\,A_t = K_t$ for all t and hence T does not depend on the subsequence, that is $T = w - lim_{s\to\infty} K_s$ and also $T \in \mathcal{E}(\tau\,;\mathcal{J})$ since the unit ball of $\mathcal{E}(\tau\,;\mathcal{J})$ is weakly closed.

On the other hand if $K \in \mathcal{K}(\tau\,;\,\mathcal{J}^{(0)})$ then $|\|\mathrm{A}_{\mathrm{s}}K - K\|| \to 0$ as $\mathrm{s} \to \infty$ and also $|\|KA_s - K\|| \to 0$ as $s \to \infty$ (replace K by K^{**}). We have

$$T'(K)A_t = \lim_{s\to\infty} T'(KA_s)A_t = \lim_{s\to\infty} KA_sT''(A_t) = \lim_{s\to\infty} KA_sT'(A_s)A_t$$
$$= \lim_{s\to\infty} KT'(A_s)A_s A_t = KT A_t$$

and since this holds for all $t \in \mathbb{N}$ we have $T'(K) = KT$. This then gives $T'(A_t)K = A_tT\,KA_sK = A_tT''(K)$ and hence $T''(K) = T\,K$.

Uniqueness of T follows from $\mathcal{K}(\tau\,;\mathcal{J}^{(0)}) \supset \mathcal{R}(\mathcal{H})$. The converse, that $T \in \mathcal{E}(\tau\,;\mathcal{J})$ gives rise to a multiplier, is a consequence of Lemma (2.3.9).

We pass now to duality. Recall from the theory of normed ideals ([137], [139]) that given norming function Φ there is conjugate norming function Φ^* so that the sual of the banach space $(\wp_\Phi^{(0)}, |\,|_\Phi)$ is $(\wp_{\Phi^*}, |\,|_{\Phi^*})$ under the duality $(X, Y) \to Tr(XY)$ for $(X, Y) \in \wp_\Phi^{(0)}, \times \wp_{\Phi*}$ (we leave out of the discussion the case of $\wp_\Phi^{(0)} = \ell_1$, where the dual is $\mathcal{B}(\mathcal{H})$).

Proposition(2.3.11)[132]: Let Φ be a norming function so that $k_\Phi(\tau) = 0$, let Φ^* Be its conjugate and assume $\wp_\Phi^{(0)} \neq \ell_1$. then the dual of $\mathcal{K}\left(\tau; \wp_\Phi^{(0)}\right)$ can be identified isometrically with $(\ell_1 \times (\wp_{\Phi^*})^n)/\mathcal{N}$ where

$$\mathcal{N} = \left\{\left(\sum_{1\le j\le n} [T_j, y_j], (y_j)_{1\le j\le n}\right) \in \ell_1 \times (\wp_{\Phi^*})^{\mathrm{n}} \,\middle|\, (y_j)_{1\le j\le n} \in (\wp_{\Phi^*})^{\mathrm{n}} with \sum_{1\le j\le n} [T_j, y_j] \in \ell_1\right\}$$

and the duality map $\mathcal{K}\left(\tau; \wp_\Phi^{(0)}\right) \times (\ell_1 \times (\wp_{\Phi^*})^{\mathrm{n}}) \to \mathbb{C}$ is

$$\left(K, \left(x, (y_j)_{1\le j\le n}\right)\right) \to Tr\left(Kx \sum_{1\le j\le n} [T_j, k] y_j\right)$$

and the norm on $(\ell_1 \times (\wp_{\Phi^*})^{\mathrm{n}})$ is

$$\left(x, (y_j)_{1\le j\le n}\right) = max\left(|x|_1, \sum_{1\le j\le n} |y_j|_{\Phi^*}\right).$$

Corollary (2.3.12)[370]: Let $G_j = G_j^* \in \mathcal{E}(\tau\,;\mathcal{J})$ be such that $\sum_j |||G_j - \frac{3}{2}I||| \le 1$. Then $G_j^{1/2} \in \mathcal{E}(\tau\,;\mathcal{J})$ and there is a universal constant C, so that

$$\sum_j \left|\left|\left|\left[G_j^{1/2}, \mathrm{X}\right]\right|\right|\right| \le \sum_j \mathrm{C}|||[G_j, \mathrm{X}]||| \text{ if } \mathrm{X} \in \mathcal{E}(\tau\,;\mathcal{J})$$

and

$$\sum_j |[G_j^{1/2}, \tau]|_{\mathcal{J}} \le \sum_j \mathrm{C}|[G_j, \tau]|_{\mathcal{J}}.$$

Proof: The Lemma is an easy consequence of the functional calculus formula

$$\sum_j G_j^{1/2} = (2\pi \mathrm{i})^{-1} \int_\Gamma \sum_j (zI - G_j)^{-1} \mathrm{z}^{1/2} \mathrm{dz},$$

where Γ is the circle $|z - 3/2| = 5/4$, and of the fact that for $z \in \Gamma$ we have :

$$\sum_j |||(zI - G_j)^{-1}|||$$
$$= \frac{4}{5}\sum_j |||(zI - G_j)^{-1}(4/5(z - 3/2)I - 4/5(G_j - 3/2I))^{-1}|||$$
$$\leq \left(1 - \frac{4}{5}\right)^{-1} = 5$$

and

$$\sum_j \left[\left(zI - G_j\right)^{-1}, \mathrm{X}\right] = \sum_j \left(zI - G_j\right)^{-1}[G_j, \mathrm{X}]\left(zI - G_j\right)^{-1}.$$

Chapter 3
Regularity Properties and Strongly Self-Absorbing C^*-Algebras

We report the program to classify separable amenable C^*-algebras. Our emphasis is on the newly apparent role of regularity properties such as finite decomposition rank, strict comparison of positive elements, and $\mathcal{Z}$-stability, and on the importance of the Cuntz semigroup.

Section (3.1): Classification Program for Separable Amenable C^*-Algebras

Rings of bounded operators on Hilbert space were first studied by Murray and von Neumann in the 1930s. These rings, later called von Neumann algebras, came to be viewed as a subcategory of a more general category, namely, C^*-algebras. (The C^*-algebra of compact operators appeared for perhaps the first time when von Neumann proved the uniqueness of the canonical commutation relations.) A C^*-algebra is a Banach algebra A with involution $x \longmapsto x^*$ satisfying the C^*-algebra identity:

$$\|xx^*\| = \|x\|^2, \forall x \in A.$$

Every C^*-algebra is isometrically $*$-isomorphic to a norm-closed sub $*$-algebra of the $*$-algebra of bounded linear operators on some Hilbert space, and so may still be viewed as a ring of operators on a Hilbert space.

In 1990, the first-named initiated a program to classify amenable norm-separable C^*-algebras via K-theoretic invariants. The graded and (pre-)ordered group $K_0 \oplus K_1$ was suggested as a first approximation to the correct invariant, as it had already proved to be complete for both approximately finite-dimensional (AF) algebras and approximately circle (A$\mathbb{T}$) algebras of real rank zero ([161], [163]). It was quickly realised, however, that more sensitive invariants would be required if the algebras considered were not sufficiently rich in projections. The program was refined, and became concentrated on proving that Banach algebra K-theory and positive traces formed a complete invariant for simple separable amenable C^*-algebras.

Recent examples based on the pioneering work of Villadsen have shown that the classification program must be further revised. Two things are now apparent: the presence of a dichotomy among separable amenable C^*-algebras dividing those algebras which are classifiable via K-theory and traces from those which will require finer invariants, and the possibility—the reality, in some cases—that this dichotomy is characterised by one of three potentially equivalent regularity properties for amenable C^*-algebras.

We give a brief account of the activity in the classification program over the past decade, with particular emphasis on the now apparent role of regularity properties. After reviewing the successes of the program so far, we will cover the work of Villadsen on rapid dimension growth AH algebras, the examples of Rørdam which have necessitated the present re-evaluation of the classification program, and some results of Winter obtained in the presence of the aforementioned regularity properties. We also discuss the possible consequences for the classification program of including the Cuntz semigroup as part of the invariant (as a refinement of the k_0 and tracial invariants).

We denote by $\mathcal{K}$ the C*-algebra of compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$.For a C*-algebra A,welet $M_n(A)$ denote the algebra of $n \times n$ matrices with entries from A. The cone of positive elements of A will be denoted by A_+.

The Elliott invariant of a C*-algebra A is the 4-tuple

$$\text{Ell}(A) := \left((K_0A, K_0A^+, \textstyle\sum_A), K_1A, T^+A, \rho_A\right), \tag{1}$$

where the K-groups are the Banach algebra ones, K_0A^+ is the image of the Murray-von Neumann semigroup V(A) under the Grothendieck map, $\sum_A$ is the subset of K_0A corresponding to projections in A, T^+A is the space of positive tracial linear functionals on A,and ρA is the natural pairing of T^+A and K_0A given by evaluating atrace at aK_0-class. See Rørdam [195]. In the case of a unital C*-algebra the invariant becomes

$$\left((K_0A, K_0A^+, [1_A]), K_1A, TA, \rho_A\right)$$

where $[1_A]$is the K_0-class of the unit, and TA is the (compact convex) space of tracial states. We will concentrate on unital C* -algebras in the sequel in order to limit technicalities.

The original statement of the classification conjecture for simple unital separable amenable C*-algebras read as follows:

Conjecture(3.1.1)[145]: Let Aand B be simple unital separable amenable C*-algebras, and suppose that there exists an isomorphism

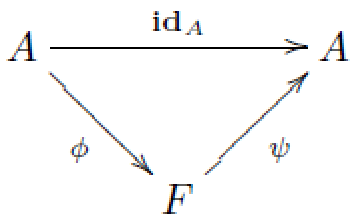
$$\emptyset : \text{Ell}(A) \to \text{Ell}(B).$$

It follows that there is $a\ *-$isomorphism Φ: A $\to$ B which inducs $\emptyset$.

It will be convenient to have an abbreviation for the statement above. Let us call it (EC).

We will take the following deep theorem, which combines results of Choi and Effros ([153]), Connes ([155]), Haagerup ([174]), and Kirchberg ([180]), to be our definition of amenability.

Theorem (3.1.2)[145]: A C*algebra A is amenable if and only if it has the following property: for each finite subset G of A and $\varepsilon > 0$ there are a finite-dimensional C*-algebra F and completely positive contractions $\emptyset, \psi$ such that the diagram



commutes up to ϵ on G .

The property characterising amenability in Theorem (3.1.2) is known as the completely positive approximation property.

Why do we consider only separable and amenable C^*-algebras in the classification program? It stands to reason that if one has no good classification of the weak closures of the GNS representations for a class of C^* -algebras, then one can hardly expect to classify the C^* -algebras themselves. These weak closures have separable predual if the C^*-algebra is separable. Connes and Haagerup gave a classification of injective von Neumann algebras with separable predual (see [156] and [175]), while Choi and Effros established that a C^*-algebra is amenable if and only if the weak closure in each GNS representation is injective ([154]). Separability and amenability are thus natural conditions which guarantee the existence of a good classification theory for the weak closures of all GNS representations of a given C^* -algebra. The assumption of amenability (at least for a simple classification; cf. however [164]) has been shown to be necessary by Dadarlat ([160]), see Rørdam [195].

One of the three regularity properties alluded to is defined in terms of the Cuntz semigroup, an analogue for positive elements of the Murray-von Neumann semigroup $V(A)$. It is known that this semigroup will be a vital part of any complete invariant for separable amenable C^*-algebras ([201]). We present both its original definition, and a modern version which makes the connection with classical K-theory more transparent.

Definition(3.1.3)[145]:(Cuntz-Rørdam; see [158] and [199]). Let $M_\infty(A)$ denote the algebraic limit of the direct system $(M_n(A), \emptyset_n)$, where $\emptyset_n : M_n(A) \to M_{n+1}(A)$is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Let $M_\infty(A)_+$ (resp. $M_n(A)_+$) denote the positive elements in $M_\infty(A)$(resp. $M_n(A)$) Given $a, b \in M_\infty(A)_+$ we say that a is Guntz subequivalent to b (written $a \precsim b$) if there is a sequence of $(v_n)_{n=1}^{\infty}$ elements in some $M_k(A)$such that

$$\|v_n b v_n^* - a\| \xrightarrow{n\to\infty} 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$)if $a \precsim b$ and $a \precsim b$.This relation is an equivalence relation, and we write $\langle a\rangle$ for the equivalence class of a.The set

$$W(A) := M_\infty(A)_+/\sim$$

becomes a (positive) ordered Abelian semigroup when equipped with the operation

$$\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$$

and the partial order

$$\langle a\rangle \le \langle b\rangle \Leftrightarrow a \precsim b.$$

Definition(3.1.3) is slightly unnatural, as it fails to consider positive elements in $A\otimes\mathcal{K}$. This defect is the result of mimicking the construction of the Murray-von Neumann semigroup. Each projection in $A\otimes\mathcal{K}$ is equivalent to a projection in some $M_n(A)$, whence $M_\infty(A)$ is large enough to encompass all possible equivalence classes of projections. The same is not true, however, of positive elements and Cuntz equivalence. The definition below amounts essentially to replacing $M_\infty(A)$ with $A\otimes\mathcal{K}$ in the definition above (this is a theorem), and also gives a new and very useful characterisation of Cuntz subequivalence, see [183] and [188].

Consider A as a (right) Hilbert C^*-module over itself, and let H_A denote the countably infinite direct sum of copies of this module. There is a one-to-one correspondence between closed countably generated submodules of H_A and hereditary subalgebras of $A\otimes\mathcal{K}$ the hereditary subalgebra B corresponds to BH_A.Since A is separable, B is singly hereditarily generated, and it is fairly routine to prove that any two generators are Cuntz equivalent in the sense of Definition (3.1.3). Thus, passing from positive elements to Cuntz equivalence classes factors through the passage from positive elements to the hereditary subalgebras they generate.

Let X and Y be closed countably generated submodules of H_A. Recall that the compact operators on H_A form a C^*-algebra isomorphic to $A \otimes \mathcal{K}$. Let us say that X is compactly contained in Y if X is contained in Y and there is a compact self-adjoint endomorphism of Y which fixes X pointwise. Such an endomorphism extends naturally to a compact self-adjoint endomorphism of H_A,and so may be viewed as a self-adjoint element of $A \otimes \mathcal{K}$. Let us write $X \precsim Y$ if each closed countably generated compactly contained submodule of$X$is isomorphic to such a submodule of Y.

Theorem(3.1.4)[145]:(Coward-Elliott-Ivanescu, [157]). *The* relation $\precsim$ on Hilbert C^*-modules defined above, when viewed as a relation on positive elements in $\mathrm{M}_\infty(\mathrm{A})$, is precisely the relation $\precsim$ of Definition (3.1.3)

Let [X] denote the Cuntz equivalence class of the module X.One may construct a positive ordered Abelian semigroup $\mathrm{Cu}(\mathrm{A})$ by endowing the set of countably generated Hilbert C^*-modules over A with the operation

$$[X] + [Y] := [X \oplus \mathrm{Y}]$$

and the partial order

$$[\mathrm{X}] \leq [\mathrm{Y}] \iff \mathrm{X} \precsim \mathrm{Y}.$$

The semigroup $\mathrm{Cu}(A)$coincides with $W(A)$ whenever A is stable, i.e., $\mathrm{A} \otimes \mathcal{K} \cong \mathrm{A}$, and has some advantages over $W(A)$ in general. First, suprema of increasing sequences always exist in $\mathrm{Cu}(A)$. This leads to the definition of a category including this structure in which $\mathrm{Cu}(A)$ sits as an object, and as a functor into which it is continuous with respect to inductive limits. (Definition(3.1.3) casts $\mathrm{W}(\mathrm{A})$ asa functor into just the category of ordered Abelian semigroups with zero. This functor fails to be continuous with respect to inductive limits.) Second, it results in the simplification in the case that A has stable rank one that Cuntz equivalence of positive elements amounts simply to isomorphism of the corresponding Hilbert C^*-modules. This has led, via recent work of Brown, Perera, to the complete classification of all countably generated Hilbert C^*-modules over A via K_0 and traces, provided that A has the relatively common property of strict comparison ([150], [151]), and to the classification of closed unitary orbits of positive operators in $\mathrm{A} \otimes \mathcal{K}$ through recent work of Ciuperca ([152]).

Cuntz equivalence is often described roughly as the Murray-von Neumann equivalence of the support projections of positive elements. This heuristic is, modulo accounting for projections, precise in C^*-algebras for which the Elliott invariant is known to be complete ([192]). In the stably finite case, one recovers K_0,the tracial simplex, and the pairing p (see (1)) from the Cuntz semigroup, whence the invariant

$$(Cu\,(A), K_1 A)$$

is finer than Ell(A) in general. These two invariants determine each other in a natural way for the largest class of unital stably finite C^*-algebras in which (EC) can be expected to hold ([150], [151]). The class in question consists of those algebras which satisfy a certain regularity property, instability, which we shall introduce presently.

We describe three agreeable properties which a C^*-algebra may enjoy. We will see later that virtually all classification theorems for separable amenable C^*-algebras via the Elliott invariant assume, either explicitly or implicitly, one of these properties.

The first regularity property—strict comparison—is one that guarantees, in simple C^*-algebras, that the heuristic view of Cuntz equivalence described is in fact accurate for positive elements which are not Cuntz equivalent to projections (see [192]). The property is K-theoretic in character.

Let A be a unital C^*-algebra, and denote by $QT(A)$ the space of normalised 2-quasitraces on A (v. [147]). Let $S(W(A))$ denote the set of additive and order preserving maps d from $W(A)$to $\mathbb{R}^+$ having the property that $d(\langle 1_A \rangle) = 1$. Such maps are called states.Given $\tau \in QT(A)$, one may define a map $d_\tau : M_\infty(A)_+ \to \mathbb{R}^+$ by

$$d_\tau(a) = \lim_{n=\infty} \tau\left(a^{\frac{1}{n}}\right). \tag{2}$$

This map is lower semicontinuous, and depends only on the Cuntz equivalence class of a. It moreover has the following properties:

(i) if $a \precsim b$, then $d_\tau(a) \leq d_\tau(b)$;

(ii) if a and b are orthogonal, then $d_\tau\,(a + b) = d_\tau\,(a) + d_\tau\,(b)$.

Thus, d_τ defines a state on $W(A)$. Such states are called lower semicontinuous dimension functions, and the set of them is denoted by $LDF(A)$. If A has the property that $a \precsim b$ whenever $d(a) < d(b)$ for every $d \in LDF(A)$, then let us say that A has strict comparison of positive elements or simply strict comparison.

A theorem of Haagerup asserts that every element of $QT(A)$ is in fact a trace if A is exact ([176]). All amenable C^*-algebras are exact, so we dispense with the consideration of quasi-traces from here on.

The second regularity property, introduced by Kirchberg and Winter, is topological in flavour. It is based on a noncommutati veversion of covering dimension called decomposition rank.

Definition (3.1.5)[145]: ([182], Definitions 2.2 and 3.1). Let A be a separable C^*-algebra.

(i) We shall say that a completely positive map $\varphi : \bigoplus_{i=1}^{s} M_{r_i} \to A$ is n- decomposable if there is a decomposition $\{1, \ldots, s\} = \coprod_{j=0}^{n} I_j$ such that the re-striction of φ to $\bigoplus_{i \in I_i} . M_{r_i}$ preserves orthogonality for each $j \in \{0, \ldots, n\}$.

(ii) A will be said to have decomposition rank n, denoted by $dr\,A = n$, if n is the least integer such that the following holds: Given $\{b_1, \ldots, b_m\} \subset A$ and $\varepsilon > 0$, there is a completely positive approximation (F, ψ, φ) for $b_1, \ldots, b_m$ within (ϵ i.e., $\psi : A \to F$ and $\varphi : F \to A$ are completely positive contractions and $\|\varphi\psi(b_i) - b_i\| < \epsilon$) such that φ is n -decomposable. If no such n exists, we write $dr\,A = \infty$.

Decomposition rank has good permanence properties. It behaves well with respect to quotients, inductive limits, hereditary subalgebras, unitization and stabilization. Its topological flavour comes from the fact that it generalises covering dimension in the commutative case: if X is a locally compact second countable space, then $dr C_0(X) = \dim X$, see [182].

The regularity property that we are interested in is finite decomposition rank, expressed by the inequality $dr < \infty$. This can only occur in a stably finite C^*-algebra.

The Jiang-Su algebra Z is a simple separable amenable and infinite-dimensional C^*-algebra with the same Elliott invariant as $\mathbb{C}$ ([177]). We say that a second algebra A is Z-

stable if A $\otimes$Z $\cong$ A. Z-stability is our third regularity property. It is very robust with respect to common constructions (see[205]).

The next theorem shows Z-stability to be highly relevant to the classification program. Recall that a pre-ordered Abelian group (G, G^+)issaid tobe weakly unperforated if $nx \in G^+\backslash\{0\}$ implies $x \in G^+$ for any $x \in G$ and $n \in \mathbb{N}$.

Theorem(3.1.6)[145]: (Gong-Jiang-Su, [173]). Let A be a simple unital C^*-algebra with weakly unperforated K_0-group. It follows that

$$Ell(A) \cong Ell(A \otimes Z).$$

Thus, in the setting of weakly unperforated K_0, the completeness of $Ell(\bullet)$inthe simple unital case of the classification program would imply Z-stability. Remarkably, there exist algebras satisfying the hypotheses of the above theorem which are
not Z-stable ([196], [201], [202]).

In general, no two of the regularity properties above are equivalent. The most important general result connecting them is the following theorem of M. RØrdam ([197]):

Theorem (3.1.7)[145]: Let A be a simple, unital, exact, finite, and Z -stable C^*-algebra. Then, A has strict comparison ofpositive elements.

We shall see later that for a substantial class of simple, separable, amenable, and stably finite C^*-algebras, all three of our regularity properties are equivalent. The algebras in this class which do satisfy these three properties also satisfy (EC). There is good reason to believe that the equivalence of these three properties will hold in much greater generality, at least in the stably finite case; in the general case, strict comparison and Z-stability may well prove to be equivalent characterisations of those simple, unital, separable, and amenable C^*-algebras which satisfy (EC).

We have two goals to edify with the classification program and to demonstrate that the regularity properties of pervade the known confirmations of (EC). This is a new point of view, for when these results were originally proved, there was no reason to think that any thing more than simplicity, separability, and amenability would be required to complete the classification program.

We have divided our review of known classification results into three broad categories according to the types of algebras covered: purely infinite algebras, and two formally different types of stably finite algebras. We will thus choose, from each of the three categories above, the classification theorem with the broadest scope, and indicate how the algebras it covers satisfy at least one of our regularity properties.

We first consider a case where the theory is summarised with one beautiful result. Recall that a simple separable amenable C^*-algebra is purely infinite if every non-zero hereditary subalgebra contains an infinite projection (a projection is infinite if it is equivalent, in the sense of Murray and von Neumann, to a proper subprojection of itself; otherwise the projection is finite).

Theorem (3.1.8)[145]: (Kirchberg-Phillips, 1995, [179] and [193]). Let A and B be separable amenable purely infinite simple C^*-algebras which satisfy the Universal Coefficient Theorem. If there is an isomorphism

$$\emptyset: \text{Ell(A)} \rightarrow \text{Ell(B)},$$

then there is a * - isomorphism Φ: A → B which induces

In the theorem above, the Elliott invariant is somewhat simplified. The hypotheses on A and B guarantee that they are traceless, and that the order structure on K_0 is irrelevant. Thus, the invariant is simply the graded group $K_0 \oplus K_1$,along with the K_0-class of the unit if it exists. The assumption of the Universal Coefficient Theorem (UCT) is required in order to deduce the theorem from a result which is formally more general: A and B as in the theorem are *-isomorphic if and only if they are KK-equivalent. The question of whether every amenable C^*-algebra satisfies the UCT is open.

Which of our three regularity properties are present here? As noted earlier, finite decomposition rank is out of the question. The algebras we are considering are traceless, and so the definition of strict comparison reduces to the following statement: for any two non-zero positive elements a, b ∈ A,wehave a $\precsim b$.This, in turn, is often taken as the very definition of pure infiniteness, and can be shown to be equivalent to the definition preceding Theorem(3.1.8)without much difficulty. Strict comparison is thus satisfied in a slightly vacuous way. As it turns out, A and B are also instable, although this is less obvious. One first proves that A and B are approximately divisible (again, this does not require Theorem (3.1.8)) and then uses the fact, due to Winter, that any separable and approximately divisible C^*-algebra is z-stable ([206]).

We now move on to the case of stably finite C^*-algebras, i.e., those algebras A such that that every projection in the (unitization of) each matrix algebra $M_n(A)$ is finite. (The question of whether a simple amenable C^*-algebra must always be purely infinite or stably finite was recently settled negatively by Rørdam. We will address his example again later.) Many of the classification results in this setting apply to classes of C^*-algebras which can be realised as inductive limits of certain building block algebras. The original classification result for stably finite algebras is due to Glimm. Recall that a C^*-algebra A is uniformly hyperfinite (UHF) if it is the limit of an inductive sequence

$$M_{n_1} \xrightarrow{\emptyset_1} M_{n_2} \xrightarrow{\emptyset_2} M_{n_3} \xrightarrow{\emptyset_3} \ldots,$$

where each $\emptyset_i$ is a unital $*$_homomorphism. We will state his result here as a confirmation of the Elliott conjecture, but note that it predates both the classification program and the realisation that K-theory is the essential invariant.

Theorem(3.1.9)[145]: (Glimm, 1960, [170]). Let A and B be UHF algebras, and suppose that there is an isomorphsim

$$\emptyset: \text{Ell(A)} \rightarrow \text{Ell(B)}.$$

It follows that there is a $*$_isomorphism Φ: A → B which induces ∅

Again, the invariant is dramatically simplified here. Only the ordered K_0-group is non-trivial. The strategy of Glimm's proof (which did not use K-theory explicitly) was to "intertwine" two inductive sequences $(M_{n_i}, \emptyset_i)$and (M_{m_i}, ψ_i).e., to find sequences of *-homomorphisms η_i and γ_i the diagramcommute. One then gets an isomorphism between the limit algebras by extending the obvious morphism between the induct ive sequences by continuity.

$$\begin{array}{ccccccc} M_{n_1} & \xrightarrow{\phi_1} & M_{n_2} & \xrightarrow{\phi_2} & M_{n_3} & \xrightarrow{\phi_3} & \cdots \\ \downarrow \gamma_1 & \nearrow \eta_1 & \downarrow \gamma_2 & \nearrow \eta_2 & \downarrow \gamma_3 & \nearrow \eta_3 & \\ M_{m_1} & \xrightarrow{\psi_1} & M_{m_2} & \xrightarrow{\psi_2} & M_{m_3} & \xrightarrow{\psi_3} & \cdots \end{array}$$

The intertwining argument above can be pushed surprisingly far. One replaces the inductive sequences above with more general inductive sequences $(A_i, \emptyset_i)$ and (B_i, ψ_i), where the A_i and B_i are drawn from a specified class (matrix algebras over circles, for instance), and seeks maps η_i and γ_i as before. Usually, it is not possible to find η_i and γ_i making the diagram commute, but approximate commutativity on ever larger finite sets can be arranged, and this suffices for the existence of an isomorphism between the limit algebras. This generalised intertwining is known as the Elliott Intertwining Argument.

The most important classification theorem for inductive limits covers the so-called approximately homogeneous (*AH*) algebras. An *AH* algebra A is the limit of an inductive sequence (A_i, ϕ_i), where each A_i is semi-homogeneous:

$$A_i = \bigoplus_{j=1}^{n_i} p_{ij}\left(C(X_{i,j}) \otimes \mathcal{K}\right) p_{i,j}$$

for some natural number n_i, compact metric spaces X_{ij}, and projections $P_{i,j} \in C(X_{ij}) \otimes \mathcal{K}$. We refer to the sequence $(A_i, \emptyset_i)$ a s a decomposition for A;such decompositions are not unique. All *AH* algebras are separable and amenable.

Let A be a simple unital *AH* algebra. Let us say that A has slow dimension growth if it has a decomposition $(A_i, \emptyset_i)$ satisfyi

$$\lim_{i\to\infty} \sup \sup \left\{ \frac{\dim(X_{i,1})}{rank\ (p_{i,1})}, \dots, \frac{\dim(X_{i,n_i})}{rank\ (p_{i,n_i})} \right\} = 0$$

Let us say that A has very slow dimension growth if it has a decomposition satisfying the (formally) stronger condition that

$$\lim_{i\to\infty} \sup \sup \left\{ \frac{\dim(X_{i,1})^3}{rank\ (p_{i,1})}, \dots, \frac{\dim(X_{i,n_i})^3}{rank\ (p_{i,n_i})} \right\} = 0$$

Finally, let us say that A has bounded dimension if there is a constant $M > 0$ and a decomposition of A satisfying

$$\lim_{i,l} \{dim(X_{i,l})\} \leq M.$$

Theorem(3.1.10)[145]: (Elliott-Gong, Dadarlat, and Gong, [166], [159] and [172]). *(EC)* holds among simple unital *AH* algebras with slow dimension growth and real rank zero.

Theorem(3.1.11)[145]: (Elliott-Gong-Li and Gong, [168] and [171]). (EC) holds among simple unital AH algebras with very slow dimension growth.

All three of our regularity properties hold for the algebras of Theorems (3.1.10) and (3.1.11), but some are easier to establish than others. Let us first point out that an algebra from either class has stable rank one and weakly unperforated K_0-group (cf. [146]), and that these facts predate Theorems (3.1.10) and (3.1.11). A simple unital C^*-algebra of real rank zero and stable rank one has strict comparison if and only if its K_0-group is weakly unperforated (cf. [191]), whence strict comparison holds for the algebras covered by Theorem (3.1.10) A recent result shows that strict comparison holds for any simple unital AH algebra with slow dimension growth ([204]), and this result is independent of the classification theorems above. Thus, strict comparison holds for the algebras of Theorems (3.1.10) and (3.1.11), and the proof of this fact, while not easy, is at least much less complicated than the proofs of the classification theorems themselves. Establishing finite decomposition rank requires the full force of the classification theorems: a consequence of both theorems is that the algebras they cover are all in fact simple unital AH algebras of bounded dimension, and such algebras have finite decomposition rank by [37, Corollary 3.12 and 3.3 (ii)]. Proving i-stability is also an application of Theorems (3.1.8) and (3.1.11): one may use these theorems to prove that the algebras in question are approximately divisible ([167]), and this entails i-stability for separable C^*-algebras ([206]).

Why all the interest in inductive limits? Initially at least, it was surprising to find that any classification of C^*-algebras by K-theory was possible, and the earliest theorems to this effect considered inductive limits (see AF algebras and AT-algebras of real rank zero in [161] and [163], respectively; it should be pointed out that [161] is based not only on [170] but on the generalisation of Glimm's approach to the full class of AF algebras by Bratteli in [148]—in which even the class of AF algebras is mentioned for the first time). But it was the realisation by Evans that a very natural class of C^*-algebras arising from dynamical systems—the irrational rotation algebras— were in fact inductive limits of elementary building blocks that began the drive to classify inductive limits of all stripes ([165]). This theorem of Elliott and Evans has recently been generalised in sweeping fashion by Lin and Phillips, who prove that virtually every C^*-dynamical system giving rise to a simple algebra is an inductive limit of fairly tractable building blocks ([187]). This result continues to provide strong motivation for the study of inductive limit algebras.

Natural examples of separable amenable C^*-algebras are rarely equipped with obvious and useful inductive limit decompositions. Even the aforementioned theorem of Lin and Phillips, which gives an inductive limit decomposition for each minimal C^*-dynamical system, does not produce inductive sequences covered by existing classification theorems. It is thus desirable to have theorems confirming the Elliott conjecture under hypotheses that are (reasonably) straightforward to verify for algebras not given as inductive limits.

Lin in [184] introduced the concept of tracial topological rank for C^*-algebras. His definition, is this: a unital simple tracial C^*-algebra A has tracial topological rank at most $n \in N$ if for any finite set FC A, tolerance $\varepsilon > 0$, and positive element $a \in A$ there exist unital subalgebras B and C of A such that

(i) $1_A = 1_B \oplus 1_C$,
(ii) $\mathcal{F}$ is almost (to within e) contained in $B \oplus C$,
(iii) is isomorphic to $F \otimes C(X)$, where $\dim(X) \leq n$ and $\mathcal{F}$ is finite-dimensional,and
(iv) 1_B is dominated, in the sense of Cuntz subequivalence, by a.

One denotes by $TR(A)$ the least integer n for which A satisfies the definition above; this is the tracial topological rank, or simply the tracial rank, of A.

The most important value of the tracial rank is zero. Lin proved that simple unital separable amenable C^*-algebras of tracial rank zero satisfy the Elliott conjecture, modulo the ever present UCT assumption ([185]). The great advantage of this result is that its hypotheses can be verified for a wide variety of C^*-dynamical systems and all simple non-commutative tori, without ever having to prove that the latter have tractable inductive limit decompositions (see [194]). Indeed, the existence of such decompositions is a consequence of Lin's theorem! (Rather, it is a consequence of his proof, which showed that his class coincided with that of

[166].) One can also verify the hypotheses of Lin's classification theorem for many real rank zero C^*-algebras with unique trace ([149]), always with the assumption, indirectly, of strict comparison.

Simple unital C^* -algebras of tracial rank zero can be shown to have stable rank one and weakly unperforated K_0-group, whence they have strict comparison of positive elements by a theorem of Perera ([191]). (There is a classification theorem for algebras of tracial rank one ([186]), but this has been somewhat less useful—it is difficult to verify tracial rank one in natural examples. Also, Niu has recently proved a classification theorem for some C^*-algebras which are approximated in trace by certain subalgebras of $M_n \otimes C[0,1]$ ([189], [190]).)

And what of our regularity properties? Lin proved in [184] that every unital simple C^*-algebra of tracial rank zero has stable rank one and weakly unperforated K_0 -group. These facts, entail strict comparison and are not nearly so difficult to prove as the tracial rank zero classification theorem. In a further analogy with the case of AH algebras, finite decomposition rank and z-stability can only be verified by applying Lin's classification theorem—aconsequence of this theorem (or rather, its proof; cf. above) is that the algebras it covers are in fact AH algebras of bounded dimension!

Until the mid 1990s we had no examples of simple separable amenable C^*-algebras where one of our regularity properties failed. To be fair, two of our regularity properties had not yet even been defined, and strict comparison was seen as a technical version of the more attractive Second Fundamental Comparability Question for projections (this last condition, abbreviated $FCQ2$, asks for strict comparison for projections only). This all changed when Vil-ladsen produced a simple separable amenable and stably finite C^*-algebra which did not have $FCQ2$, answering a long-standing question of Blackadar ([208]). The techniques introduced by Villadsen were subsequently used by him and others to answer many open questions in the theory of nuclear C^*-algebras including the following:

(i) Does there exist a simple separable amenable C^*-algebra containing a finite and an infinite projection? (Solved affirmatively by RØrdam in [196].)

(ii) Does there exist a simple and stably finite C^*-algebra with non-minimal stable rank? (Solved affirmatively by Villadsen in [209].)

(iii) Is stability a stable property for simple C^*-algebras? (Solved negatively by Rørdam in [198].)

(iv) Does a simple and stably finite C^*-algebra with cancellation of projections necessarily have stable rank one?

(v) Are the C^*-algebras of minimal dynamical systems always classified by their Elliott invariants? (Solved negatively by Kerr and Giol in [178].)

Of the results above, (i) was (and is) the most significant. In addition to showing that simple separable amenable C^*-algebras do not have a factor-like type classification, Rørdam's example demonstrated that the Elliott invariant as it stood could not be complete in the simple case. This and other examples have necessitated a revision of the classification program ([203]).

(EC) does not hold in general, and this justifies new assumptions in efforts to confirm it. In particular, one may assume any combination of our three regularity properties. We will comment on the aptness of these new assumptions. For now we observe that, from a certain point of view, we have been making these assumptions all along. Existing classification theorems for stably finite C^*-algebras of real rank zero are accompanied by the crucial assumptions of stable rank one and weakly unperforated K_0; as has already been pointed out, unperforated K_0 can be replaced with strict comparison in this setting.

How much further can one get by assuming the (formally) stronger condition of Z-stability? What role does finite decomposition rank play? As it turns out, these two properties both alone and together produce interesting results. Let 72.72.0 denote the class of simple unital separable amenable C^*-algebras of real rank zero. The following subclasses of 770 tysatisfy (EC):

(i) algebras that satisfy the UCT, have finite decomposition rank, and have tracial simplex with compact and zero-dimensional extreme boundary;

(ii)Z-stable algebras that satisfy the UCT and arapproximated locally by subalgebras of finite decomposition rank.

These results, due to Winter ([210], [211]), showcase the power of our regularity properties: included in the algebras covered by (ii) are all simple separable unital Z-stable ASH (approximately subhomogeneous) algebras of real rank zero.

Another advantage to the assumptions of Z-stability and strict comparison is that they allow one to recover extremely fine isomorphism invariants for C^*-algebras from the Elliott invariant alone. (This recovery is not possible in general.) We will be able to give precise meaning to this comment below, but first require a further discussion of the Cuntz semigroup.

A natural reaction to an incomplete invariant is to enlarge it: include whatever information was used to prove incompleteness. This is not always a good idea. It is possible that one's distinguishing information is ad hoc and unlikely to yield a complete invariant. Worse, one may throw so much new information into the invariant that the impact of its potential completeness is severely diminished.

Rørdam's finite-and-infinite-projection example is distinguished from a simple and purely infinite algebra with the same K-theory by the obvious fact that the latter contains no finite projections. The natural invariant which captures this difference is the semigroup of Murray-von Neumann equivalence classes of projections in matrices over an algebra A, denoted by $V(A)$. After the appearance of Rørdam's example, the second-named author produced a pair of simple, separable, amenable, and stably finite C^*-algebras which agreed on the Elliott invariant but were not isomorphic. In this case the distinguishing invariant was Rieffel's stable rank. It was later discovered that these algebras could not be distinguished by their Murray-von Neumann semigroups, but it was not yet clear which data were missing from the Elliott invariant. More dramatic examples were needed, ones which agreed on most candidates for enlarging the invariant and pointed the way to the "missing information".

[202] constructed a pair of simple unital AH algebras which, while non-isomorphic, agreed on a wide swath of invariants including the Elliott invariant, all continuous (with respect to inductive sequences) and homotopy invariant functors from the category of C^*-algebras (a class which includes the Murray-von Neumann semigroup), the real and stable ranks, and, as was shown later in [203], stable isomorphism invariants (those invariants which are insensitive to tensoring with a matrix algebra or passing to a hereditary subalgebra). It seemed reasonable to expect that the distinguishing invariant in this example—the Cuntz semigroup—might be a good candidate for enlarging the invariant. At least, it was an object which after years of being used sparingly as a means to other ends, merited study for its own sake.

Let us collect some evidence supporting the addition of the Cuntz semigroup to the usual Elliott invariant. First, in the biggest class of algebras where (EC) can be expected to hold—Z-stable algebras, as shown by Theorem (3.1.6)—it is not an addition at all! Recent work of Brown, Perera shows that for a simple unital separable amenable C^*-algebra which absorbs Z tensorially, there is a functor which recovers the Cuntz semigroup from the Elliott invariant ([150], [192]). This functorial recovery also holds for simple unital AH algebras of slow dimension growth, a class for which Z-stability is not known and yet confirmation of (EC) is expected. (It should be noted that the computation of the Cuntz semigroup for a simple approximate interval (AI) algebra was essentially carried out by Ivanescu and the first-named author in [169], although one does require [12, Corollary 4] to see that the computation is complete.)

Second, the Cuntz semigroup unifies the counterexamples of Rørdam. One can show that the examples of [195], [201], and [202] all consist of pairs of algebras with different Cuntz semigroups; there are no known counterexamples to the conjecture that simple separable amenable C^*-algebras will be classifiedupto *-isomorphism by the Elliott invariant and the Cuntz semigroup.

Third, the Cuntz semigroup provides a bridge to the classification of non-simple algebras. Ciuperca and the first-named author have recently proved that AI algebras—limits of inductive sequences of algebras of the form

$$\bigoplus_{i=1}^{n} M_{m_i}(C[0,1])$$

are classified up to isomorphism by their Cuntz semigroups ([152]). This is accomplished by proving that the approximate unitary equivalence classes of positive operators in the unitization of a stable C^*-algebra of stable rank one are determined by the Cuntz semigroup of the algebra, and then appealing to a theorem of Thom-sen ([200]). (These approximate unitary equivalence classes of positive operators can be endowed with the structure of a metric Abelian semigroup with functional calculus. This invariant, known as Thomsen's semigroup, is recovered functorially in [152] from the Cuntz semigroup for a C^*-algebra of stable rank one, and so from the Elliott invariant in an algebra which is moreover simple, unital, exact, finite, and Z-stable by the results of [150].

There is one last reason to suspect a deep connection between the classification program and the Cuntz semigroup. Let us first recall a theorem of Kirchberg, which is germane to the classification of purely infinite C^*-algebras cf. Theorem (3.1.8).

Theorem(3.1.12)[145]: (Kirchberg, c. 1994; see [181]). Let A be a separable amenable C^*-algebra. The following two properties are equivalent:

(i) A is purely infinite;
(ii)A $\otimes \mathcal{O}_\infty \cong$ A.

A consequence of Kirchberg's theorem is that among simple separable amenable C^*-algebras which merely contain an infinite projection, there is a two-fold characterisation of the (proper) subclass which satisfies the original form of the Elliott conjecture (modulo UCT). If one assumes apriori that A is simple and unital with no tracial state, then a theorem of Rørdam (see [197]) shows that the property (ii) above, known as $\mathcal{C}_\infty$-stability, is equivalent to i-stability. Under these same hypotheses, the property (i) is equivalent to the statement that A has strict comparison. Kirchberg's theorem can thus be rephrased as follows in the simple unital case:

Theorem(3.1.13)[145]: Let A be a simple separable unital amenable C^* -algebra without a tracial state. The following two properties are equivalent:

(i) A has strict comparison;
(ii)A $\otimes$Z $\cong$ A.

The properties (i) and (ii) in the theorem above make perfect sense in the presence of a trace. We moreover have that (ii) implies (i) even in the presence of traces (this is due to Rørdam; see [197]). It therefore makes sense to ask whether the theorem might be true without the tracelessness hypothesis. Remarkably, this appears to be the case. Winter have proved that for a substantial class of stably finite C^*-algebras, strict comparison and i-stability are equivalent, and that these properties moreover characterise the (proper) subclass which satisfies (EC) ([207]). In other words, Kirchberg's theorem is quite possibly a special case of a more general result, one which will give a unified two-fold characterisation of those simple separable amenable C^*-algebras which satisfy the original form of the Elliott conjecture.

It is too soon to know whether the Cuntz semigroup together with the Elliott invariant will suffice for the classification of simple separable amenable C^*-algebras, or indeed, whether such a broad classification can be hoped for at all. But there is already cause for optimism. Zhuang Niu has recently obtained some results on lifting maps at the level of the Cuntz semigroup to $*$_homomorphisms. This type of lifting result is a key ingredient in proving almost any kind of classification theorem (cf. [164]). His results suggest the algebras of [202] as the appropriate starting point for any effort to establish the Cuntz semigroup as a complete isomorphism invariant, at least in the absence of K_1 (see [152]).

Section (3.2): Z-Stable Wilhelm Winter

A separable unital C^*-algebra $\mathcal{D} \neq \mathbb{C}$ is called strongly self-absorbing, if there is an isomorphism $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to The first factor embedding, cf. [222]. The interest in such algebras largely arises From Elliott’s program to classify nuclear C^*-algebras by K-theoretic invariants.

Examples suggest that classification will only be possible up to $\mathcal{D}$-stability (i.e., up to tensoring with $\mathcal{D}$) for a strongly self-absorbing $\mathcal{D}$, cf. [221], [145], [224]. While the known strongly self-absorbing examples are quite well understood, and are entirely classified, it remains an open problem whether these are the only ones. Froma more general perspective, the question is in how far abstract properties allow for comparison with concrete examples. For nuclear C^*-algebras, this question prominently manifests itself as the UCT problem (i.e., is every nuclear C^*-algebra KK-equivalent to a commutative one); a positive answer even in the special setting of strongly self-absorbing C^*-algebras would be highly satisfactory, and likely shed light on the general case.

We shall be concerned with a closely related interpretation of the aforementioned question: we will show that any strongly self-absorbing C^* -algebra $\mathcal{D}$ admits a unital embedding of a specific example, the Jiang–Su algebra $\mathcal{Z}$ (see [177] and to [220] for an introduction and various characterizations of $\mathcal{Z}$). It then follows immediately that $\mathcal{D}$ is in fact $\mathcal{Z}$-stable. The result answers some problems left open in [222] and in [213]; in particular it implies that strongly self-absorbing C^*-algebras are always K_1-injective. It shows that the Jiang–Su algebra is an initial object in the category of strongly self-absorbing C^*-algebras (with initial *-homomorphisms); there can only be one such initial object, whence $\mathcal{Z}$ is characterized this way. It is interesting to note that the Cuntz algebra $\mathcal{O}_2$ is the uniquely determined final object in this category, and that $\mathcal{O}_\infty$ can be characterized as the initial object in the category of infinite strongly self-absorbing C^*-algebras.

The Proof of the main result builds on ideas from [220] and from [213], where the problem was settled in the case where $\mathcal{D}$ contains a nontrivial projection.

We generalize a technical result from [213] to a setting that does not require the existence of projections, see Lemma (3.2.4) below. See [219] for a brief account of the Cuntz semigroup.

Proposition(3.2.1)[212]: Let A be a unital C*-algebra, $0 \le g \le 1_A$.
Then, for any $0 \ne n \in \mathbb{N}$, we have

$$\begin{aligned} 1_{A^{\otimes n}} - g^{\otimes n} \ge\ & (1_A - g)\otimes g \otimes \ldots \otimes g \\ & + g \otimes (1_A - g) \otimes g \otimes \ldots \otimes g \\ & \vdots \\ & + g \otimes \ldots \otimes g \otimes (1_A - g). \end{aligned}$$

Proof: The statement is trivial for $n = 1$. Suppose now we have shown the assertion for some $0 \ne n \in \mathbb{N}$. We obtain

$$\begin{aligned} 1_{A^{\otimes(n+1)}} - g^{\otimes(n+1)} &= 1_{A^{\otimes n}} \otimes g - g^{\otimes n} \otimes g + 1_{A^{\otimes n}} \otimes (1_A - g) \\ &= \left(1_{A^{\otimes n}} - g^{\otimes n}\right) \otimes g + 1_{A^{\otimes n}} \otimes (1_A - g) \\ &\ge \left((1_A - g) \otimes g \otimes \ldots \otimes g\right) \otimes g \\ &\quad + (g \otimes (1_A - g) \otimes g \otimes \ldots \otimes g) \otimes g \\ &\quad \vdots \\ &\quad + \left(g \otimes \ldots \otimes g \otimes (1_A - g)\right) \otimes g \\ &\quad + g^{\otimes n} \otimes (1_A - g), \end{aligned}$$

where for the inequality we have used our induction hypothesis as well as the fact that $1_{A^{\otimes n}} \otimes (1_A - g) \ge g^{\otimes n} \otimes (1_A - g)$. Therefore, the statement also holds for $n + 1$.

Proposition(3.2.2)[212]: Let $\mathcal{D}$ be strongly self-absorbing, $0 \le d \le 1_{\mathcal{D}}$.
Then, for any $0 \ne k \in \mathbb{N}$,

$$[1_{\mathcal{D}^{\otimes k}} - d^{\otimes k}] \le k \cdot [(1_{\mathcal{D}} - d) \otimes 1_{\mathcal{D}^{\otimes (k-1)}}] \; in \; W(\mathcal{D}^{\otimes k})$$

Proof: The assertion holds trivially for $k = 1$. Suppose now it has been verified for some $k \in \mathbb{N}$. Then,

$$\begin{aligned}[1_{\mathcal{D}} - d^{\otimes (k+1)}] &= [1_{\mathcal{D}^{\otimes k}} \otimes (1_{\mathcal{D}} - d) + 1_{\mathcal{D}^{\otimes k}} \otimes d - d^{\otimes k} \otimes d] \\ &\le [1_{\mathcal{D}^{\otimes k}} \otimes (1_{\mathcal{D}} - d)] + [(1_{\mathcal{D}^{\otimes k}} - d^{\otimes k}) \otimes 1_{\mathcal{D}}] \\ &\le [(1_{\mathcal{D}} - d) \otimes 1_{\mathcal{D}^{\otimes k}}] + k \cdot [(1_{\mathcal{D}} - d) \otimes 1_{\mathcal{D}^{\otimes (k-1)}} \otimes 1_{\mathcal{D}}] \\ &= (k+1) \cdot [(1_{\mathcal{D}} - d) \otimes 1_{\mathcal{D}^{\otimes k}}]\end{aligned}$$

(using that $\mathcal{D}$ is strongly self-absorbing as well as our induction hypothesis for the second inequality), so the assertion also holds for $k + 1$.

The following is only a mild generalization of [213, Lemma 1.3].

Lemma(3.2.3)[212]: Let $\mathcal{D}$ be strongly self-absorbing and let $0 \le f \le g \le 1_{\mathcal{D}}$ be positive elements of $\mathcal{D}$ satisfying $1_{\mathcal{D}} - g \neq 0$ and $fg = f$.
Then, there is $0 \neq n \in \mathbb{N}$ such that

$$[f^{\otimes n}] \le [1_{\mathcal{D}^{\otimes k}} - g^{\otimes n}] \text{in } W(\mathcal{D}^{\otimes n}).$$

Proof: Since $\mathcal{D}$ is simple and $1_{\mathcal{D}} - g \neq 0$, there is $n \in \mathbb{N}$ such that

$$[f] \le n \cdot [1_{\mathcal{D}} - g].$$

Then,

$$\begin{aligned}[f^{\otimes n}] &\le n \cdot [(1_{\mathcal{D}} - g) \otimes f \otimes \ldots \otimes f] \\ &= [(1_{\mathcal{D}} - g) \otimes f \otimes \ldots \otimes f] + \ldots + [f \otimes \ldots \otimes f \otimes (1_{\mathcal{D}} - g)] \\ &= [(1_{\mathcal{D}} - g) \otimes f \otimes \ldots \otimes f + \ldots + f \otimes \ldots \otimes f \otimes (1_{\mathcal{D}} - g)] \\ &\le [(1_{\mathcal{D}} - g) \otimes g \otimes \ldots \otimes g + \ldots + g \otimes \ldots \otimes g \otimes (1_{\mathcal{D}} - g)] \\ &\le [1_{\mathcal{D}^{\otimes n}} - g^{\otimes n}],\end{aligned}$$

where for the first equality we have used that $\mathcal{D}$ is strongly self-absorbing, for the second equality we have used that the terms are pairwise orthogonal by our assumptions on f and g, and the last inequality follows from Proposition (3.2.1).

The following is a version of [213] for positive elements rather than projections.

Lemma(3.2.4)[212]: Let $\mathcal{D}$ be strongly self-absorbing and let $0 \le f \le g \le 1_{\mathcal{D}}$ be positive elements satisfying $1_{\mathcal{D}} - g \neq 0$ and $fg = f$; let $0 \neq d \in \mathcal{D}_+$.
Then, there is $0 \neq m \in \mathbb{N}$ such that

$$[f^{\otimes m}] \le [d \otimes 1_{\mathcal{D}^{\otimes (m-1)}}] \text{ in } W(\mathcal{D}^{\otimes m})$$

Proof: By Lemma (3.2.3), there is $0 \neq n \in \mathbb{N}$ such that

$$[f^{\otimes n}] \le [1_{\mathcal{D}^{\otimes n}} - g^{\otimes n}];$$

since $f^{\otimes n} \perp 1_{\mathcal{D}^{\otimes n}} - g^{\otimes n}$, this implies

$$2 \cdot [f^{\otimes n}] \le [1_{\mathcal{D}^{\otimes n}}].$$

By an easy induction argument we then have

$$2^k \cdot [f^{\otimes nk}] \le [1_{\mathcal{D}^{\otimes nk}}]$$

for any $k \in \mathbb{N}$.

By simplicity of $\mathcal{D}$ and since d is nonzero, there is $\bar{k} \in \mathbb{N}$ such that

$$[f] \le 2^{\bar{k}} \cdot [d].$$

Set

$$m := n\bar{k} + 1,$$

then

$$[f^{\otimes m}] \le 2^{\bar{k}} \cdot [d \otimes f^{\otimes(m-1)}] = 2^{\bar{k}} \cdot [d \otimes f^{\otimes n\bar{k}}] \le [d \otimes 1_{\mathcal{D}^{\otimes n\bar{k}}}]$$
$$= [d \otimes 1_{\mathcal{D}^{\otimes(m-1)}}].$$

Below we establish the existence of nontrivial order zero maps from matrix algebras into strongly self-absorbing C^*-algebras, and we show certain systems of such maps give rise to order zero maps with small complements, see [225] and [226].

Proposition(3.2.5)[212]: Let $\mathcal{D}$ be strongly self-absorbing and $0 \neq d \in \mathcal{D}_+$. Then, for any $0 \neq k \in \mathbb{N}$, there is a nonzero c.p.c. order zero map

$$\psi : M_k \to \overline{d\mathcal{D}d}.$$

Proof: Let us first prove the assertion in the case where $d = 1_{\mathcal{D}}$ and $k = 2$. Since $\mathcal{D}$ is infinite dimensional, there are orthogonal positive normalized elements $e, f \in \mathcal{D}$. Since $\mathcal{D} \cong D \otimes D$ is strongly self-absorbing, there is a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}\otimes\mathcal{D}$ such that

$$u_n(e \otimes f)u_n^* \xrightarrow{n\to\infty} f \otimes e;$$

since $e \otimes f \perp f \otimes e$ this implies that there is a c.p.c. order zero map

$$\bar{\sigma} : M_2 \to \prod_{\mathbb{N}} \mathcal{D}\otimes\mathcal{D}/\otimes_{\mathbb{N}}\mathcal{D}\otimes\mathcal{D}$$

given by

$$\bar{\sigma}(e_{11}) = e \otimes f, \bar{\sigma}(e_{22}) = f \otimes e, \bar{\sigma}(e_{21}) = \pi\left(\left(u_n(e\otimes f)\right)_{n\in\mathbb{N}}\right)$$

$(cf. [225])$, where $\pi : \prod_{\mathbb{N}} \mathcal{D}\otimes\mathcal{D} \to \prod_{\mathbb{N}} \mathcal{D}\otimes\mathcal{D} /\otimes_{\mathbb{N}}\mathcal{D}\otimes\mathcal{D}$ denotes the quotient map. Since order zero maps with finite dimensional domains are semi projective (cf. [225]), $\bar{\sigma}$ has a c.p.c. order zero lift $M_2 \to \prod_{\mathbb{N}} \mathcal{D}\otimes\mathcal{D}$ which in turn implies that there is a nonzero c.p.c. order zero map

$$\bar{\sigma} : M_2 \to \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D} \cong \mathcal{D}.$$

Next, if $k = 2^r$ for some $r \in \mathbb{N}$, then

$$M_{2^r} \cong (M_2)^{\otimes r} \xrightarrow{\tilde{\sigma}^{\otimes r}} \mathcal{D}^{\otimes r} \cong \mathcal{D}$$

is a nonzero c.p.c. order zero map; for an arbitrary $k \in \mathbb{N}$, we may take r large enough and restrict $\tilde{\sigma}^{\otimes r}$to $M_k \subset M_{2^r}$ to obtain a nonzero c.p.c. order zero maps : $\sigma: M_k \to \mathcal{D}$. This settles the proposition for arbitrary k and for $d = 1_{\mathcal{D}}$. Now if d is an arbitrary nonzero positive element (which we may clearly assume to be normalized), we can define a c.p.c. map

$$\bar{\psi} : M_k \to \prod_{\mathbb{N}} \overline{d\mathcal{D}d}/\otimes_{\mathbb{N}} \overline{d\mathcal{D}d} \subset \prod_{\mathbb{N}} \mathcal{D} /\otimes_{\mathbb{N}} \mathcal{D}$$

by setting

$$\bar{\psi}(x) := \pi((d\sigma_n(x)d)_{n\in\mathbb{N}}) \text{ for } x \in M_k,$$

where again $\pi : \prod_{\mathbb{N}} \overline{d\mathcal{D}d} \to \prod_{\mathbb{N}} \overline{d\mathcal{D}d}/\otimes_{\mathbb{N}} \overline{d\mathcal{D}d}$ denotes the quotient map and $\sigma_n: M_k \to \mathcal{D}$ is a sequence of c.p.c. maps lifting the c.p.c. order zero map

$$\mu\sigma : M_k \to \left(\prod_{\mathbb{N}} \mathcal{D}/\otimes_{\mathbb{N}} \mathcal{D}\right) \cap \mathcal{D}',$$

with

$$\mu : \mathcal{D} \to \left(\prod_{\mathbb{N}} \mathcal{D}/\otimes_{\mathbb{N}} \mathcal{D}\right) \cap \mathcal{D}'$$

being a unital *-homomorphism as in [222, Theorem 2.2]. It is straightforward to check that $\bar{\psi}$ is nonzero and has order zero. Again by semiprojectivity of order zero maps, this implies the existence of a nonzero c.p.c. order zero map

$$\psi: M_k \to \overline{d\mathcal{D}d}$$

Proposition(3.2.6)[212]: Let B be a unital *-algebra and $\varrho: M_2 \to B$ a unital *-homomorphism. Define

$$E := \{f \in C([0,1], B \otimes M_2) \mid f(0) \in B \otimes 1_{M_2}, f(1) \in 1_B \otimes M_2\}.$$

Then, there is a unital *-homomorphism

$$\tilde{\varrho}: M_2 \to E$$

Proof: This follows from simply connecting the two embeddings $\varrho \otimes 1_{M_2}$ And $1_{M_2} \otimes \mathrm{id}_{M_2}$ of M_2 into $\varrho(M_2) \otimes M_2 \cong M_2 \otimes M_2$ M along the unit interval.

Lemma(3.2.7)[212]: Let $m \in \mathbb{N}$ and A a unital C^*-algebra. Let

$$\varphi_1, \ldots, \varphi_m: M_2 \to A$$

be c.p.c. order zero maps such that

$$\sum_{i=1}^{m} \varphi_i(1_{M_2}) \leq 1_A$$

and

$$[\varphi_i(M_2), \varphi_j(M_2)] = 0 \text{ if } i \neq j.$$

Then, there is a c.p.c. order zero map

$$\bar{\varphi}: M_2 \to C^*(\varphi_i(M_2) \mid i = 1, \ldots, m) \subset A$$

such that

$$\bar{\varphi}(1_{M_2}) = \sum_{i=1}^{m} \varphi_i(1_{M_2}).$$

Moreover, if $d \in A_+$satisfies $\varphi_m(e_{11})d = d$,we may assume that $\bar{\varphi}(e_{11})d = d$.

Proof: In the following, we write $C_i, i = 1, \ldots, m$, for various copies of the C^*-algebra $C_0((0,1], M_2)$; these come equipped with c.p.c. order zero maps $\varrho_i: M_2 \to C_i$ given by

$$\varrho_i(x)(t) = t \cdot x \text{ for } t \in (0,1] \text{ and } x \in M_2.$$

By [223, Proposition 3.2(a)], the c.p.c. order zero maps $\varphi_i: M_2 \to A$ induce unique *-homomorphisms $C_i \to A$ via $\varrho_i(x) \mapsto \varphi_i(x)$, for $x \in M_2$.

We now define a universal C^*-algebra

$$B := C^*(C_i, 1 \mid \sum_{l=1}^{m} \varrho_l(1_{M_2}) \leq 1, [C_i, C_j] = 0 \text{ if } i \neq j \in \{1, \ldots, m\}).$$

Then, B is generated by the $\varrho_i(x), i \in \{1, \ldots, m\}$ and $x \in M_2$; the assignment $\varrho_i(x) \mapsto \varphi_i(x)\ for\ i \in \{1, \ldots, m\}$ and $x \in M_2$

induces a unital *-homomorphism

$$\pi: B \to C^*, (\varphi_i(M_2), 1_A \mid i \in \{1, \ldots, m\}) \subset A$$

satisfying

$$\sum_{l=1}^{m} \pi\varrho_l(1_{M_2}) = \sum_{l=1}^{m} \varphi_l(1_{M_2})$$

Now if we find a c.p.c. order zero map

$$\bar{\varrho} : M_2 \to B$$

Satisfying

$\bar{\varrho}(1_{M_2}) = \sum_{l=1}^{m} \varrho_l(1_{M_2})$,

Then

$$\bar{\varphi} := \pi\bar{\varrho}$$

will have the desired properties, proving the first assertion of the lemma. We proceed to construct $\bar{\varrho}$.

For $k = 1, \dots, m$, let

$$J_k := \mathcal{J}\,(1 - \sum_{l=k}^{m} \varrho_l(1_{M_2})) \lhd B$$

denote the ideal generated by $1 - \sum_{l=k}^{m} \varrho_l(1_{M_2})$ in B; let

$$B_k := B/J_k$$

denote the quotient. We clearly have

$$J_1 \subset J_2 \subset \cdots \subset J_m$$

and surjections

$$B \overset{\pi_1}{\to} B_1 \overset{\pi_2}{\to} \dots \overset{\pi_m}{\longrightarrow} B_m.$$

Observe that

$$\pi_m o \dots o\, \pi_1 o \varrho_m : M_2 \to B_m$$

is a unital surjective c.p. order zero map, hence a *-homomorphism by [223, Proposition 3.2(b)]; therefore, $B_m \cong M_2$.

For $k = 1, \dots, m - 1$, set

(i) $E_k := \{f \in C([0,1], B_{k+1} \otimes M_2) \mid f(0) \in B_{k+1} \otimes 1_{M_2}, f(1) \in 1_{B_{k+1}} \otimes M_2\}$;

one easily checks that the maps

$$\sigma_k : B_k \longrightarrow E_k$$

induced by

$$\pi_k \dots \pi_1 \sigma_i(x) \longmapsto (t \longmapsto (1 - t) \cdot \pi_{k+1} \dots \pi_1\, \sigma_i\,(x) \otimes 1M_2) \quad \text{for } i = k + 1, \dots, m \text{ and } x \in M_2$$

$$\left(t \longmapsto t \cdot 1_{B_{k+1}} \otimes x\right) \quad \text{for } i = k \text{ and } x \in M_2$$

are well-defined *-isomorphisms. Similarly, the map

$$\sigma_0 : B \to E_0 := \{f \in C([0,1], B_1) \mid f(1) \in \mathbb{C} \cdot 1_{B_1}\}$$

induced by

$$\varrho_i(x) \longmapsto \left(t \longmapsto (1 - t) \cdot \pi_1 \varrho_i\,(x)\right) \text{for } i = 1, \dots, m \text{ and } x \in M_2,$$

$$1_B \longmapsto 1_{E_0}$$

is a well-defined *-isomorphism; note that

$$\sigma_0\left(\sum_{l=1}^{m} \varrho_1\,(1M_2)\right) = (\,t \longmapsto (1 - t).\, 1B_1).$$

By (i) together with Proposition (3.2.6) and an easy induction argument, the unital *-homomorphism

$$\pi_m \dots \pi_1 \varrho_m : M_2 \to B_m$$

pulls back to a unital *-homomorphism

$$\tilde{\varrho} : M_2 \to B_1;$$

This in turn induces a c.p.c. order zero map

$$\tilde{\varrho}: M_2 \to B_0$$

By

$$\tilde{\varrho}(x) := \left(t \mapsto (1-t).\tilde{\varrho}(x)\right);$$

note that this map satisfies

$$\tilde{\varrho}\left(1_{M_2}\right) = \left(t \mapsto (1-t).1_{B_1}\right).$$

We now define a *-homomorphism note that $\tilde{\varrho}(1_{M_2}) = \sum_{l=1}^{m} \varrho_l\ (1_{M_2})$, whence $\bar{\varrho}$ is as desired.

For the second assertion of the lemma, note that $\bar{\varrho}$ and ϱ_m agree modulo J_m. Therefore,$\bar{\varphi} = \pi\bar{\varrho}$ and $\varphi_m = \pi\varrho_m$ agree up to $\pi(J_m)$. However, one checks that $\pi(J_m) \perp d$, whence $(\bar{\varphi}(x) - \varphi_m(x))d = 0$ for all $x \in M_2$. This implies $\bar{\varphi}(e_{11})d = \varphi_m(e_{11})d = d$.

Proposition (3.2.8)[212]: Let $\mathcal{D}$ be strongly self-absorbing, $0 \neq m \in \mathbb{N}$ and

$$\varphi_0: M_2 \to \mathcal{D}$$

a c.p.c. order zero map.

Then, there are c.p.c. order zero maps

$$\varphi_1, \ldots, \varphi_m: M_2 \to \mathcal{D}^{\otimes m}$$

such that

(i) $\varphi_1 = \varphi_0 \otimes 1_{\mathcal{D}^{\otimes(m-1)}}$

(ii) $\left[\varphi_1 = (M_2), \varphi_j(M_2)\right] = 0\ if\ i \neq j$

(iii) $1_{\mathcal{D}^{\otimes m}} \sum_{i=1}^{m} \varphi_i\ (1_{M_2}), = \left(1_{\mathcal{D}} - \varphi_0(1_{M_2})\right)^{\otimes m}$.

Proof: For $k \in \{1, \ldots, m\}$, define

$$\varphi_k := (\left(1_{\mathcal{D}} - \varphi_0\left(1_{M_2}\right)\right)^{\otimes(k-1)} \otimes \varphi_0 \otimes 1_{\mathcal{D}^{\otimes(m-k)}},$$

them the φ_k obviously satisfy Proposition (3.2.8)(i) and (ii).

A simple induction argument shows that, for $k = 1, \ldots, m$,

$$1_{\mathcal{D}^{\otimes m}} - \sum_{i=1}^{k} \varphi_i\ (1_{M_2}), = \left(1_{\mathcal{D}} - \varphi_0(1_{M_2})\right)^{\otimes k} \otimes 1_{\mathcal{D}^{\otimes(m-k)}},$$

which is Proposition (3.2.8)(iii) when we take $k = m$.

We now assemble the techniques of the preceding and a result from [220] to prove the main result; we also derive some consequences.

Theorem(3.2.9)[212] Any strongly self-absorbing C^*-algebra $\mathcal{D}$ absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially.

Proof: Let $k \in \mathbb{N}$. By Proposition(3.2.5), there is a nonzero c.p.c. order zero map $\varphi : M_2 \to \mathcal{D}$. Using functional calculus for order zero maps (cf. [226]), we may assume that there is

$$2 \leq d \leq \varphi(e_{11})$$

such that

$$d \neq 0 \text{ and } \varphi(e_{11})d = d.$$

Note that

$$\left(1_{\mathcal{D}} - \varphi\left(1_{M_2}\right)\right)(1_{\mathcal{D}} - d) = 1_{\mathcal{D}} - \varphi\left(1_{M_2}\right).$$

By Proposition(3.2.5), there is a nonzero c.p.c. order zero map

$$\psi : M_k \to \overline{d\mathcal{D}d};$$

note that

$$\varphi(e_{11})\psi:(x)=\psi(x) \text{ for } j = 1,\dots,k \text{ and } x \in M_k.$$

Apply Lemma(3.2.4) (with $\mathcal{D}^{\otimes k}, \psi(e_{11})^{\otimes k}, (1\mathcal{D}-\varphi(1_{M_2}))\otimes 1_{\mathcal{D}k^{\otimes(k-1)}}$ and $(1_{\mathcal{D}}-d)\otimes 1_{\mathcal{D}k^{\otimes(k-1)}}$ in place of $\mathcal{D}, d, f$ and g, respectively) to obtain $0 \neq m \in \mathbb{N}$ such that

(ii) $[((1_{\mathcal{D}}-\varphi(1_{M_2}))\otimes 1_{\mathcal{D}^{\otimes(k-1)}})^{\otimes m}] \leq [\psi(e_{11})^{\otimes k}\otimes 1_{\mathcal{D}^{\otimes(m-1)}}]$ in $W((\mathcal{D}\otimes k)^{\otimes m})$.

From Proposition (3.2.8) (with $\mathcal{D}^{\otimes k}$ in place of $\mathcal{D}$ and $\varphi_0 := \varphi\otimes 1_{D\otimes(k-1)}$) we obtain c.p.c. order zero maps

$$\varphi_1,\dots,\varphi_m: M_2 \to (\mathcal{D}^{\otimes k})^{\otimes m}$$

Satisfying (3.2.8) (i), (ii) and (iii). By relabeling the φ_i we may assume that actually $\varphi_m = \varphi_0 \otimes 1_{(\mathcal{D}^{\otimes k})^{\otimes(m-1)}}$ in (3.2.8) (i).

From Lemma (3.2.7), we obtain a c.p.c. order zero map

$$\varphi : M_2 \to C^*(\varphi_i(M_2) \mid i = 1,\dots,m) \subset (D^{\otimes k})^{\otimes m}$$

such that

$$\bar{\varphi}(1_{M_2}) = \sum_{i=1}^{m} \varphi_i(1_{M_2}).$$

By the second assertion of Lemma(3.2.7) and since

$$\varphi_m(e_{11})(\psi(1_{M_k})\otimes 1_{\mathcal{D}^{\otimes(km-1)}}) = (\varphi(e_{11})\otimes 1_{\mathcal{D}^{\otimes(km-1)}})\big(\psi(1_{M_k})\otimes 1_{\mathcal{D}^{\otimes(km-1)}}\big).$$
$$= \psi(1_{M_k})\otimes 1_{\mathcal{D}^{\otimes(km-1)}},$$

we may furthermore assume that

$$\varphi(e_{11})(\psi(1_{M_k})\otimes 1_{\mathcal{D}^{\otimes(km-1)}}) = \psi(1_{M_k})\otimes 1_{\mathcal{D}^{\otimes(km-1)}},$$

which in turn yields

(iii) $\psi(1_{M_k})\otimes 1_{\mathcal{D}^{\otimes(km-1)}} \leq \bar{\varphi}(e_{11})$

since ψ is contractive. Note that we have

$$\left[1_{(\mathcal{D}^{\otimes k})^{\otimes m}} - \bar{\varphi}(1_{M_2})\right] \overset{(3.2.8)(\text{iii})}{=} \left[\left(1_{\mathcal{D}^{\otimes k}} - \varphi_0(1_{M_2})\right)^{\otimes m}\right]$$
$$= \left[\left(1_{\mathcal{D}} - \varphi(1_{M_2})\otimes 1_{\mathcal{D}^{\otimes(m-1)}}\right)^{\otimes m}\right]$$

(iv) $\overset{(\text{ii})}{\leq} \left[\psi(e_{11})^{\otimes k}\otimes 1_{\mathcal{D}^{\otimes(km-1)}}\right]$

in $W((\mathcal{D}^{\otimes k})^{\otimes m})$. Define a c.p.c. order zero map

$$\Phi : M_{2^k} \cong (M_2)^{\otimes k} \to ((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes k} \cong \mathcal{D}^{\otimes kmk}.$$

By

$$\Phi := \bar{\varphi}^{\otimes k}.$$

We have

$$\left[1_{((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes k}} - \Phi\left(1_{(M_2)^{\otimes k}}\right)\right]$$
$$\overset{(3.2.2)}{\leq} k \cdot \left[\left(1_{(\mathcal{D}^{\otimes k})^{\otimes m}} - \bar{\varphi}(1_{M_2})\right)\otimes 1_{((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes k}}\right]$$
$$\overset{(\text{iv})}{\leq} k.\left[\psi(e_{11})^{\otimes k}\otimes 1_{(\mathcal{D}^{\otimes k})^{\otimes(m-1)}}\otimes 1_{((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes(k-1)}}\right]$$

$$\leq \left[\psi(1_{M_2})^{\otimes k} \otimes 1_{\left(\mathcal{D}^{\otimes(km-1)}\right)^{\otimes k}}\right]$$
$$\overset{\text{(iii)}}{\leq} [\bar{\varphi}(e_{11})^{\otimes k}]$$
$$= [\Phi(e_{11})]$$

In $W\left(\left((\mathcal{D}^{\otimes k})^{\otimes m}\right)^{\otimes k}\right)$. From [220] we now see that there is a unital $*$-homomorphism

$$\varrho : Z_{2^k,2^k+1} \to \mathcal{D}^{\otimes kmk} \cong \mathcal{D}.$$

Since k was arbitrary, by [206] this implies that $\mathcal{D}$ is Z-stable.

Corollary(3.2.10)[212]: The Jiang–Su algebra is the uniquely determined (up to isomorphism) initial object in the category of strongly self-absorbing C^*-algebras (with unital $*$-homomorphisms).

Proof: By Theorem (3.2.9), the Jiang–Su algebra does embed unitally into any strongly self-absorbing C^*-algebra, so it is an initial object. If $\mathcal{D}$ is another initial object, then Z and $\mathcal{D}$ embed unitally into one another, whence they are isomorphic by [222].

Sometimes an object in a category is called initial only if there is a unique morphism to any other object; this remains true in our setting if one takes approximate unitary equivalence classes of unital $*$-homomorphisms as morphisms, see [222], [214], [215], [216], [217] and [218].

Corollary (3.2.11)[370]: Let A_r be a unital C*-algebra, $0 \leq g_r \leq 1_{A_r}$.

Then, for any $0 \neq n \in \mathbb{N}$, we have

$$\sum_r \left(1_{A_r^{\otimes n}} - g_r^{\otimes n}\right) \geq \sum_r (1_{A_r} - g_r) \otimes g_r \otimes \ldots \otimes g_r$$
$$+ \sum g_r \otimes (1_{A_r} - g_r) \otimes g_r \otimes \ldots \otimes g_r$$
$$\vdots$$
$$+ \sum g_r \otimes \ldots \otimes g_r \otimes (1_{A_r} - g_r).$$

Proof: The statement is trivial for $n = 1$. Suppose now we have shown the assertion for some $0 \neq n \in \mathbb{N}$. We obtain

$$\sum_r \left(1_{A_r^{\otimes(n+1)}} - g_r^{\otimes(n+1)}\right) = \sum_r \left(1_{A_r^{\otimes n}} \otimes g_r - g_r^{\otimes n} \otimes g_r + 1_{A_r^{\otimes n}} \otimes (1_{A_r} - g_r)\right)$$
$$= \sum_r \left(\left(1_{A_r^{\otimes n}} - g_r^{\otimes n}\right) \otimes g_r + 1_{A_r^{\otimes n}} \otimes (1_{A_r} - g_r)\right)$$
$$\geq \sum_r \left((1_{A_r} - g_r) \otimes g_r \otimes \ldots \otimes g_r\right) \otimes g_r$$
$$+ \sum (g_r \otimes (1_{A_r} - g_r) \otimes g_r \otimes \ldots \otimes g_r) \otimes g_r$$
$$\vdots$$
$$+ \sum \left(g_r \otimes \ldots \otimes g_r \otimes (1_{A_r} - g_r)\right) \otimes g_r$$
$$+ g_r^{\otimes n} \otimes (1_{A_r} - g_r),$$

where for the inequality we have used our induction hypothesis as well as the fact that $\sum_r \ 1_{A_r^{\otimes n}} \otimes (1_{A_r} - g_r) \geq \sum_r \ g_r^{\otimes n} \otimes (1_{A_r} - g_r)$. Therefore, the statement also holds for $n + 1$.

Corollary(3.2.12)[370]: Let $\mathcal{D}_r$ be strongly self-absorbing, $0 \leq d_r \leq 1_{\mathcal{D}}$.
Then, for any $0 \neq k \in \mathbb{N}$,

$$\sum_r [1_{\mathcal{D}^{\otimes k}} - d_r^{\otimes k}] \leq k \cdot \sum_r [(1_{\mathcal{D}_r} - d_r) \otimes 1_{\mathcal{D}_r^{\otimes (k-1)}}] \; in \, W\left(\mathcal{D}_r^{\otimes k}\right)$$

Proof: The assertion holds trivially for $k = 1$. Suppose now it has been verified for some $k \in \mathbb{N}$. Then,

$$\sum_r \left[1_{\mathcal{D}_r} - d_r^{\otimes (k+1)}\right] = \sum_r \left[1_{\mathcal{D}_r^{\otimes k}} \otimes \left(1_{\mathcal{D}_r} - d_r\right) + 1_{\mathcal{D}_r^{\otimes k}} \otimes d_r - d_r^{\otimes k} \otimes d_r\right]$$

$$\leq \sum_r [1_{\mathcal{D}_r^{\otimes k}} \otimes (1_{\mathcal{D}_r} - d_r)] + \left[\left(1_{\mathcal{D}_r^{\otimes k}} - d_r^{\otimes k}\right) \otimes 1_{\mathcal{D}_r}\right]$$

$$\leq \sum_r \left[\left(1_{\mathcal{D}_r} - d_r\right) \otimes 1_{\mathcal{D}_r^{\otimes k}}\right] + k \cdot \left[\left(1_{\mathcal{D}_r} - d_r\right) \otimes 1_{\mathcal{D}_r^{\otimes (k-1)}} \otimes 1_{\mathcal{D}_r}\right]$$

$$= \sum_r (k + 1) \cdot \left[\left(1_{\mathcal{D}_r} - d_r\right) \otimes 1_{\mathcal{D}_r^{\otimes k}}\right]$$

(using that $\mathcal{D}_r$ is strongly self-absorbing as well as our induction hypothesis for the second inequality), so the assertion also holds for $k + 1$.

Corollary(3.2.13)[370]: Let $\mathcal{D}$ be strongly self-absorbing and let $0 \leq f_r \leq g_r \leq 1_{\mathcal{D}}$ be positive elements of $\mathcal{D}$ satisfying $1_{\mathcal{D}} - g_r \neq 0$ and $\sum_r f_r g_r = \sum_r f_r$.
Then, there is $0 \neq n \in \mathbb{N}$ such that

$$\sum_r [f_r^{\otimes n}] \leq \sum_r [1_{\mathcal{D}^{\otimes k}} - g_r^{\otimes n}] \text{in } W\left(\mathcal{D}^{\otimes n}\right).$$

Proof: Since $\mathcal{D}$ is simple and $1_{\mathcal{D}} - g_r \neq 0$, there is $n \in \mathbb{N}$ such that

$$[f_r] \leq n \cdot [1_{\mathcal{D}} - g_r].$$

Then,

$$\sum_r [f_r^{\otimes n}] \leq n \cdot \sum_r [(1_{\mathcal{D}} - g_r) \otimes f_r \otimes \ldots \otimes f_r]$$

$$= \sum_r [(1_{\mathcal{D}} - g_r) \otimes f_r \otimes \ldots \otimes f_r] + \ldots + \sum [f_r \otimes \ldots \otimes f_r \otimes (1_{\mathcal{D}} - g_r)$$

$$= \sum_r [(1_{\mathcal{D}} - g_r) \otimes f_r \otimes \ldots \otimes f_r + \ldots + f_r \otimes \ldots \otimes f_r \otimes (1_{\mathcal{D}} - g_r)]$$

$$\leq \sum_r [(1_{\mathcal{D}} - g_r) \otimes g_r \otimes \ldots \otimes g_r + \ldots + g_r \otimes \ldots \otimes g_r \otimes (1_{\mathcal{D}} - g_r)]$$

$$\leq \sum_r [1_{\mathcal{D}^{\otimes n}} - g_r^{\otimes n}],$$

where for the first equality we have used that $\mathcal{D}$ is strongly self-absorbing, for the second equality we have used that the terms are pairwise orthogonal by our assumptions on f_r and g_r, and the last inequality follows from Proposition (3.2.1).

Chapter 4
Descriptive Set Theory and Unitary Equivalence

We deduce that AF algebras are classifiable by countable structures, and that a conjecture of Winter for nuclear separable simple C^*-algebras cannot be disproved by appealing to known standard Borel structures on these algebras. We study that the automorphisms of any separable C^*-algebra that does not have continuous trace are not classifiable by countable structures up to unitary equivalence.

Section (4.1): C^*-Algebra Invariants

The classification theory of nuclear separable C^*-algebras via K-theoretic and tracial invariants was initiated by G. A. Elliott ca. 1990. An ideal result in this theory is of the following type:

Let C_1 be a category of C^*-algebras, C_2 a category of invariants, and $\mathcal{F}: C_1 \to C_2$ a functor. We say that $(\mathcal{F}, C_2)$ classifies C_1 if for any isomorphism $\phi: \mathcal{F}(A) \to \mathcal{F}(B)$ there is an isomorphism $\Phi: A \to B$ such that $\mathcal{F}(\Phi) = \phi$, and if, moreover, the range of $\mathcal{F}$ can be identified.

Given $A, B \in C_1$, one wants to decide whether A and B are isomorphic. With a theorem as above in hand (see Elliott and Toms [259] or Rørdam [269]), this reduces to deciding whether $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are isomorphic; in particular, one must compute $\mathcal{F}(B)$ and $\mathcal{F}(B)$. What does it mean for an invariant to be computable? The broadest definition is available when the objects of C_1 and C_2 admit natural parameterizations as standard Borel spaces, for the computability of $\mathcal{F}(\bullet)$ then reduces to the question "Is $\mathcal{F}$ a Borel map?" The aim is to prove that a variety of C^*-algebra invariants are indeed Borel computable, and to give some applications of these results.

The main results are summarized informally below.

Theorem (4.1.1)[257]: The following invariants of a separable C^*-algebra A are Borel computable: the (unital) Elliott invariant $\mathrm{Ell}(A)$ consisting of pre-ordered K- theory, tracial functionals, and the pairing between them;

the cuntz semigroup $\mathrm{cu}(A)$;

the radius of comparison of A;

the real and stable rank of A;

the nuclear dimension of A;

the presence of Z- stability for A;

the theory $\mathrm{th}(A)$of A.

Proving that the Elliott invariant and the Cuntz semigroup are computable turn out to be the most involved tasks.

A classification problem is a pair (X, E) consisting of a standard Borel space X, the (parameters for) objects to be classified, and an equivalence relation E, the relation of isomorphism among the objects in X. In most interesting cases, the equivalence relation E is easily definable from the elements of X and is seen to be Borel or, at worst, analytic; that is certainly the case here. To compare the relative difficulty of classification problems (X, E) and (Y, F), we employ the notion of Borel reducibility:

One says that E is Borel reducible to F if there is a Borel map $\Theta: X \to Y$ with the property that

$$xEY \Leftrightarrow \Theta(\mathrm{x}) F \Theta(Y).$$

The relation F is viewed as being at least as complicated as E. The relation E is viewed as being particularly nice when F-classes are "classifiable by countable structures". Equivalently (12), the relation E is no more complicated than isomorphism for countable graphs. Theorem (4.1.1) (i) entails the computability of the pointed (pre-)ordered K_0 -group of a unital separable C^*-algebra. As isomorphism of such groups is Borel-reducible to isomorphism of countable graphs, we have the following result.

Theorem (4.1.2)[257]: A Falgebras areclassifiable by countable structures.

In order to classify nuclear separable C^*-algebras using only K-theoretic and tracial invariants, it is necessary to assume that the algebras satisfy some sort of regularity property, be it topological, homological or C^*-algebraic (see [8] for a survey). This idea is summarized in the following conjecture of Winter and the second author.

Conjecture (4.1.3)[257]: Let A be a simple unital separable nuclear and infinite-dimensional C^*-algebra. The following are equivalent:

(i) A has finite nuclear dimension;

(ii) A is Z-stable;

(iii) A has strict comparison of positive elements.

Combining the main result of [195] with that of [16] yields (i)⇒ (ii), while Rørdam proves (ii)⇒ (iii) in[270]. Partial converses to these results follow from the successes of Elliott's classification program. Here we prove the following result.

Theorem (4.1.4)[257] The classes (i)- (iii) of conjecture (4.1.3) from Borel sets.

Therefore the classes of C^*-algebras appearing in the conjecture have the same descriptive set theoretic complexity.

We recall two parameterizations of separable C^*-algebras as standard Borel spaces; establishes the computability of the Elliott invariant; We consider the computability of the Cuntz semigroup and the radius of comparison; the Appendix deal with Z-stability, nuclear dimension, the first-order theory of a C^*-algebra in the logic of metric structures, and the real and stable rank.

In [262], we introduced four parameterizations of separable C^*-algebras by standard Borel spaces and proved that they were equivalent.

Let H be a separable infinite dimensional Hilbert space and let as usual $\mathcal{B}(H)$ denote the space of bounded operators on H. The space $\mathcal{B}(H)$ becomes a standard Borel space when equipped with the Borel structure generated by the weakly open subsets. Following [265], we let

$$\Gamma(H) = \mathcal{B}(H)^{\mathbb{N}},$$

And equip this with the product Borel structure. For each $\gamma \in \Gamma(H)$ we let $C^*(\gamma)$ be the C^*-algebras generated by the sequence γ. If we identify each $\gamma \in \Gamma(H)$ with $C^*(\gamma)$, then naturally $\Gamma(H)$ parameterizes all separable C^*-algebras acting on H. Since every separable C^*-algebra is isomorphic to a C^*-subalgebra of $\mathcal{B}(H)$ this gives us a standard Borel parameterization of the category of all separable C^*-algebras. If the Hilbert space H is clear from the context we will write Γ instead of $\Gamma(H)$. We define

$$\gamma \simeq^{\Gamma} \gamma' \Leftrightarrow C^*(\gamma) \text{ is isomorphic to } C^*(\gamma').$$

Let $\mathbb{Q}(i) = \mathbb{Q} + i\mathbb{Q}$ denote the complex rationals. Following [265], let $(p_j : j \in \mathbb{N})$ enumerate the non-commutative $*$-polynomials without constant term in the formal variables $X_k, k \in \mathbb{N}$, with coefficients in $\mathbb{Q}(i)$, and for $\gamma \in \Gamma$ write $p_j(\gamma)$for the evaluation of p_j with $X_k = \gamma(k)$. Then $C^*(\gamma)$ is the norm-closure of $\{p_j(\gamma): j \in \mathbb{N}\}$. The map $\Gamma \to \Gamma: \gamma \longmapsto \hat{\gamma}$ where $\hat{\gamma}(j) = p_{j(\gamma)}$ is clearly a Borel map from Γ to Γ. If we let

$$\hat{\Gamma}(H) = \{\hat{\gamma}: \gamma \in \Gamma(H)\},$$

then $\hat{\Gamma}(H)$ is a standard Borel space and provides another parameterization of the C^*-algebras acting on H ; we suppress H and writ$\hat{\Gamma}$ whenever possible. For $\gamma \in \hat{\Gamma}$ let $\check{\gamma} \in \Gamma$ be defined by

$$\hat{\Gamma}(n) = \gamma(i) \Leftrightarrow p_i X_n$$

and note that $\hat{\Gamma} \to \Gamma: \gamma \mapsto \check{\gamma}$ is the inverse of $\Gamma \to \hat{\Gamma}: \gamma \mapsto \check{\gamma}$ we let $\simeq^{\hat{\Gamma}}$ be defined by

$$\gamma \simeq^{\hat{\Gamma}} \Leftrightarrow C^*(\gamma) \text{ is isomorphic to } C^*(\gamma').$$

It is clear from the above that Γ and $\hat{\Gamma}$ are equivalent parameterizations.
An alternative picture of $\hat{\Gamma}(H)$ is obtained by considering the free (i.e., surjectively universal) countable unnormed $\mathbb{Q}(i)$-$*$-algebra $\mathfrak{A}$. We can identify $\mathfrak{A}$ with the set $\{p_n: n \in \mathbb{N}\}$. Then

$$\hat{\Gamma}_{\mathfrak{A}}(H) = \{f: \mathfrak{A} \to \mathcal{B}(H): f \text{ is } a * \text{homomorphism}\} \text{ to } C^*(f')$$

is easily seen to be a Borel subset of $\mathcal{B}(H)^{\mathfrak{A}}$. For $f \in \hat{\Gamma}_{\mathfrak{A}}letC^*(f)$. be the norm closure of $\operatorname{im}(f)$, and define

$$f \simeq^{\hat{\Gamma}_{\mathfrak{A}}} f' \Leftrightarrow C^*(f) \text{is isomorphic } C^*(f').$$

Clearly, the map $\hat{\Gamma} \to \hat{\Gamma}_{\mathfrak{A}}: \gamma \mapsto f_\gamma$ defined by $f_\gamma(p_j) = \gamma(j)$ provides a Borel bijection witnessing that $\hat{\Gamma}$ and $\hat{\Gamma}_{\mathfrak{A}}$ are equivalent (and therefore they are also equivalent to Γ.)

If we instead consider the free countable unital unnormed $\mathbb{Q}(i)$-$*$-algebra $\mathfrak{A}_{\mathrm{u}}$ and let

$$\hat{\Gamma}_{\mathfrak{A}_u}(H) = \{f: \mathfrak{A}_{\mathrm{u}} \mathcal{B}(H): f \text{ } is \text{ unital} * \text{homomorphism}\},$$

then this gives a parameterization of all unital separable C^*-subalgebras of $\mathcal{B}(H)$. Note that $\mathfrak{A}_u$ may be identified with the set of all formal $*$-polynomials in the variables X_k with coefficients in $\mathbb{Q}(i)$ (allowing a constant term).
We introduce a standard Borel space of Elliott invariants. We prove that the computation of the Elliott invariant of $C^*(\gamma)$ is given by a Borel-measurable function. The Elliott invariant of a unital C^*-algebra A is the sextuple (see [268], [269]).

$$K_0(A), K_0(A)^+, [1_A]_0), K_1(A), T(A), r_A: T(A) \to S(K_0(A)).$$

Here, $K_0(A), K_0(A)^+, [1_A]_0)$ is the ordered K_0 -group with the canonical order unit, $K_1(A)$is the K_1 -group of A, and $T(A)$is the Choquet simplex of all tracial states of A. Recall that a state ϕ on a unital C^*-algebra A is tracial if $\phi(ab) = \phi(ba)$ for all a and b in A. Finally, $r_A: T(A) \to S(K_0(A))$ is the coupling map that ssociates a state on $K_0(A)$ to every trace on A. Recall that a state on an ordered Abelian group is a positive homomorphism $f: G \to (\mathbb{R}, +)$ and that the Murray-von Neumann equivalence of pro jections p and q in A implies $\phi(p) = \phi(q)$ for every trace ϕ on A.

As usual, identify $n \in \mathbb{N}$ with the set $\{0, 1, \ldots, n-1\}$. For $n \in \mathbb{N} \cup \{\mathbb{N}\}$, let

$$S(n) = \{f: n^2 \to n: (\forall i, j, k \in n) f(i, f(j, k)) = f(f(i, j), k)\}$$

Note that $S(\mathbb{N})$is closed when $\mathbb{N}^{\mathbb{N}^2}$ is given the product topology, and that if for $f \in S(n)$ and $i, j \in n$ we define $i \cdot_f j$ as $f(i, j)$, then $\cdot_f$ gives n a semigroup structure. The space $S(n)$may therefore be thought of as a Polish space parameterizing all countable semigroups with underlying set $n \in \mathbb{N} \cup \{\mathbb{N}\}$. We let $S'(n) = S(n) \times n$, and think of elements $(f, i) \in S'(n)$ as the space of semigroups with a distinguished element i.

The subsets of $S(n)$ (respectively $S'(n)$) consisting of A be semigroups, groups and Abelian groups form closed subspaces that we denote by $S_a(n), G(n)$ and $G_a(n)$ (respectively $S'_a(n), G'(n)$ and $G'_a(n)$.

The isomorphism relation in $S(n), S_a(n), G(n)$ and $G_a(n)$, as well as the corresponding "primed" classes, are induced by the natural action of the symmetric group Sym (n). These are very special cases of the logic actions, see [145].

We also define the spaces $G_{\text{ord}}(n)$ and$G'_{\text{ord}}(n)$ of ordered Abelian with a distinguished order unit, in the sense of Rodman [269]. The space $G_{\text{ord}}(n)$ consists of pairs $(f, X) \in G_a(n) \times \mathcal{P}(n)$ such that if we define for $x, y \in n$ the operation $x +_f y = f(x, y)$ and $x \leq xy \Leftrightarrow y +_f (-x) \in X$, then we have $X +_f X \subseteq X, -X \cap X = \{0\}$ and $X - X = n$. The space $G'_{\text{ord}}(n)$ consists of pairs $\big((f, X), u\big) \in G_{\text{ord}}(n) \times n$ satisfying additionally the conditions

(i) $u \in X$;

(ii) for all $x \in n$ there is $k \in \mathbb{N}$ such that $-ku \leq xX \leq xku$.

From their definition it is easy to verify that $G_{\text{ord}}(\mathbb{N})$ and $G'_{\text{ord}}(\mathbb{N})$ form G_δ subsets of $G_a(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ and $G_{\text{ord}}(\mathbb{N}) \times \mathbb{N}$, and so are Polish spaces.

Define G_{ord} and G'_{ord} to be the disjoint unions

$$G_{\text{ord}} = \coprod_{n \in \mathbb{N} \cup \{\mathbb{N}\}} G_{\text{ord}}(n) \text{ and } G'_{\text{ord}} = \coprod_{n \in \mathbb{N} \cup \{\mathbb{N}\}} G'_{\text{ord}}(n)$$

and give these spaces the natural standard Borel structure. Similarly, define the standard Borel spaces S, S_a, G and G_a and their primed counterparts to be the disjoint union of their respective constituents.

Recall that a compact convex set K is a Choquet simplex if for every point x in K there exists a unique probability measure μ supported by the extreme boundary of K that has x as its barycentre. Every metrizable Choquet simplex is affinely homeomorphic to a subset of $\Delta^{\mathbb{N}}$, with $\Delta = [0, 1]$.

For every C^*-algebra A the space $T(A)$ of its traces is a Choquet simplex. In case when A is separable it can be identified with a compact convex subset of the Hilbert cube $\Delta^{\mathbb{N}}$. In [262] it was shown that all Choquet simplexes form a Borel subset of the $F(\Delta^{\mathbb{N}})$).

Astate on ordered Abelian group with unit $(G, G^+, 1)$ is a homomorphism $\phi: G \to \mathbb{R}$ such that $\phi[G^+] \subseteq \mathbb{R}^+$and $\phi(1) = 1$. For every $n \in \mathbb{N} \cup \{\mathbb{N}\}$ the set $Z_0$of all $\big((f, X, u)\phi\big) \in G'_{\text{ord}}(n) \times \mathbb{R}^n$ such that $\phi[X] \subseteq \mathbb{R}^+, \phi(u) = 1$ And $\phi\big(f(i, j)\big) = \phi(i) + \phi(j)$ for all i, j is clearly closed. By [262], the map states: $G'_{\text{ord}}(n) \to \mathbb{R}^n$ such that States(f, X, u) of Z_0 at $(f, X, u)$is Borel.

Recall that $\mathbb{K}_{\text{conv}}$ denotes the compact metric space of compact convex subsets of $\Delta^{\mathbb{N}}$.

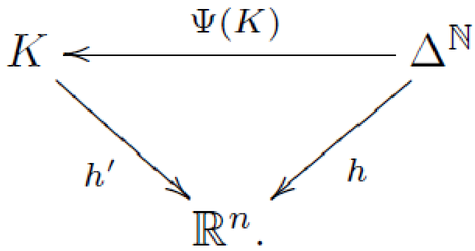
Lemma (4.1.5)[257]: There is a continuous map $\Psi: \mathbb{K}_{\text{conv}} \to C(\Delta^{\mathbb{N}}, \Delta^{\mathbb{N}})$ such that $\Psi(K)$ is a retraction from $\Delta^{\mathbb{N}}$ onto K for all $K \in \mathbb{K}_{\text{conv}}$.

Proof: Identify $\Delta^{\mathbb{N}}$ with $\prod_n [-1/n, 1/n]$ and consider the compatible ℓ_2 metric d_2 on $\Delta^{\mathbb{N}}$. Consider the set

$$Z = \left\{(K, x, y): K \in \mathbb{K}_{\text{conv}}, x \in \Delta^{\mathbb{N}}, y \in K \text{and} d_2(x, y) = \inf_{z \in K} d_2(x, z)\right\}$$

Since the map $(K, x) \mapsto \inf_{z \in K} d_2(x, z)$ is continuous on $\{K \in \mathbb{K}_{\text{conv}}: K \neq \emptyset\}$, this set is closed. Also, for every

K, x there is the unique point y such that $(K, x, y) \in Z$ (e.g., [266]). By compactness, the function x that sends (K, x) to the unique y such that $(K, x, y)(K, x, y) \in Z$ is continuous. Again by compactness, the map $\Psi\ (K)\ =\ \{\ (x, y): (K, x, y)\ \in\ Z\}$ is continuous for $K \in \mathbb{K}_{\text{conv}}, n \in \mathbb{N} \cup \{\mathbb{N}\}$, and $(f, X, u) \in G'_{\text{ord}}\ (n)$ let Pairing (f, X, u) be the set of all $h: \Delta^{\mathbb{N}} \to \mathbb{R}^n$ such that there exists a continuous affine function $h': K \to$ States (X, f, u) such that with Ψ as in Lemma (4.1.5) the following diagram commutes



Again the set of all $(K, (f, X, u), h)$ as above is closed and by [262] the map Pairing is Borel.++ G_0, G_1, T, r.

Definition (4.1.6)[257]: The space Ell of Elliott invariants is a subspace of

$$G'_{\text{ord}} \times G_a \times \mathbb{K}_{\text{conv}} \times \coprod_{n \in \mathbb{N} \cup \{\mathbb{N}\}} C(\Delta^{\mathbb{N}}.\Delta^n)$$

consisting of quadruples G_0, G_1, T, r where $G_0 \in G'_{\text{ord}}, G_1 \in G_a, T \in \mathbb{K}_{\text{conv}}$ is a Choquet simplex, and $r \in \text{Pairing}(T, G_0)$. By the above and [262], the set Ell is Borel and therefore it is a standard Borel space with the induced Borel structure.

We say that two such quadruples(G_0, G_1, T, r)and (G_0', G_1', T', r') in Ell are isomorphic if $G_0 \cong G_0', G_1 \cong G_1'$ and there is an affine isomorphism $\alpha: T \to T'$ such that we have $\hat{\eta} \restriction T \circ r = r' \circ \alpha \restriction T'$, where $\hat{\eta}: S(G_0) \to S(G_0')$ corresponds to some isomorphism $\eta: G_0 \to G_0'$. This is clearly an analytic ceequivalen relation.

The isomorphism relation defined above is clearly analytic and it corresponds to the isomorphism of Elliott invariants. The rest contains the proof of the following theorem.

We will start by showing:

For a C^*-algebra A, let $\sim_A$ denote the Murray-von Neumann equivalence of projections in A. Therefore, $p \sim_A q$ if there is $v \in A$ such that $vv^* = p$ and $v^*v = q$. Note that $p \sim_A q$ implies $\phi(p) = \phi(q)$ for every trace ϕ of A. If A is clear from the context we will simply write $\sim$. Also, following the usual conventions, for $a, p \in B(H)$ we write $a \oplus b$a for the element

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_2(B(H)).$$

For the next Lemma, recall from [262] the Borel function projections $j: \Gamma\ \to\ \Gamma$ which computes, for each $\gamma \in \Gamma$, a sequence of projections that are dense in the set of projections in $C^*(\gamma)$.

Lemma (4.1.7)[257]: (i) The relation $r_2 \subseteq \Gamma \times \mathbb{N} \times \mathbb{N}$ defined by

$$r_1(y, m, n) \Leftrightarrow proj(y)(m) \sim C^*(\gamma) proj(y)(n)$$

is Borel.

(ii) The reˆlation $Y_2 \subseteq \Gamma \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ defined by

$$r_2(y, m, n, k) \Leftrightarrow proj(y)(m) \oplus proj(y)(n) \sim M_2(C^*(y)) proj(y)(k) \oplus 0$$

is Borel.

Proof: To see (i), note that

$r_1(y,m,n) \Leftrightarrow (\exists k)\|pk(y)pk(y)^* - \mathrm{proj}(y)(m)\| < \frac{1}{4} \wedge \|pk(y)^*pk(y) - \mathrm{proj}(y)(n)\| < \frac{1}{4}$.

For (ii), note that for $m,n,k \in \mathbb{N}$ the maps $\Gamma \to M_2\big(B(H)\big)$

$$y \mapsto \mathrm{proj}(y)(m) \oplus \mathrm{proj}(y)(n) \text{and } y \mapsto \mathrm{proj}(y)(k) \oplus 0$$

are Borel by farah et al. [262]. Thus,

$$r_2(m,n,k) \Leftrightarrow (\exists i)\left\|p_i\big(M_2(y)\big)p_i\big(M_2(y)\big)^* - \mathrm{proj}(y)(m)\oplus\mathrm{proj}(y)(n)\right\| < \frac{1}{4} \wedge \left\|p_i\big(M_2(y)\big)p_i\big(M_2(y)\big)^* - \mathrm{proj}(y)(k)\oplus 0\right\| < \frac{1}{4}$$

gives a Borel definition of r_2.

Proposition (4.1.8)[257]: There is a Borel map $K_{0,u}\colon \Gamma_{\mathrm{u}} \to G'_{\mathrm{ord}}$ such that

$$K_{0,u}(\gamma) \cong \left(K_0\big(C^*(\gamma)\big)^+, [1_{C^*(y)}]_0\right)$$

for all γ.

Proof. Note in Lemma (4.1.7) that for each $y \in \Gamma_u, (r_1)_y = \{(m,n) \in \mathbb{N}\}\colon r_1(y,m,n)$ defines an equivalence relation denoted on $\mathbb{N}$. Let $B_n \subseteq \Gamma_u, (n \in \mathbb{N} \cup \{\infty\})$ be the set of $y \in \Gamma$ such that $(r_1)_y$ has exactly n classes. Then (B_n) is a Borel partition of Γ_u.. On each $B_n$we can find Borel functions $\sigma_{ni}\colon B_n \to \mathbb{N}, (0 \le \mathrm{i} < \mathrm{n})$,selecting exactly one point in each $(r_1)_y$-class. Identifying $n \in \mathbb{N}$ with the set $\{0,\dots,n-1\}$, let $V_0\ (y)$ (where $y \in B_n$) be the semigroup on n defined by

$$i + j = k \Leftrightarrow r_2\big(y, \sigma_{n,i}(y), \sigma_{n,j}(y), \big)\sigma_{n,k}(y)$$

By Farah et al. [255] there is a Borel map $\psi\colon \Gamma \to \Gamma$ such that $C^*\big(\psi(y)\big) \simeq C^*(y) \otimes \mathcal{K}$. We define $V(y) = V_0\big(\psi(y)\big)$and note that this gives us a Borel assignment $B_n \to S(n)$ of semigroup structures on n. The K_0 group of $C^*(y)$ is then the Grothendieck group constructed from $V(y)$ with the order unit being the unique $i \in n$ such that $\sigma_{n,i}(y) \sim_y u(y)$, and so the proof is complete once we prove the next Lemma.

Lemma (4.1.9)[257]: There is a Borel map $S_a \to G_{\mathrm{ord}}$ associating to each Abelian semigroup (defined by) $f \in S_a$ the Grothendieck group constructed from f.

Proof: It is enough to construct a Borel $S_a(n) \to G_{\mathrm{ord}}$ as required for each $n \in \mathbb{N} \cup \{\mathbb{N}\}$. We follow the description of the Grothendieck group given in [6]. Defining

$$P = \left\{\big(f,(i,j),(k,i)\big) \in s_a \times n^2 \times n^2 \colon (\exists m) i +_f \iota +_f + m = k +_f j +_f m\right\},$$

we have that P_f is an equivalence relation on n^2 for all $f \in S_a$. Write $S_a(n)$ as a disjoint union of Borel pieces $B_k(k \in \mathbb{N} \cup \{\mathbb{N}\})$such that $f \in B_k$ if and only if P_f has exactly k classes. We can then find on each piece B_k Borel functions selecting an element in each P_f class, and from t selecti on the Grothendieck group of f can be defined on k in a Borel way.

Corollary (4.1.10)[257]: There is a Borel map $K_0\colon \Gamma \to G_{\mathrm{ord}}$ such that

$$K_0(y) \simeq \Big(K_0\big(C^*(y)\big), K_0^+\big(C^*(y)\big)\Big).$$

Proof: By Farah et al [262] the unitization $\tilde{C}^*(y)$ of $C^*(y)$is obtained via a Borel function, and by the above proof so is $K_0\left(\tilde{C}^*(y)\right)$. Then $K_0C^*(y)$ is isomorphic to the quotient of $K_0\left(\tilde{C}^*(y)\right)$by its subgroup generated by the image of the identity in $\tilde{C}^*(y)$.

Proposition (4.1.11)[257]: There is a Borel map $K_1\colon \Gamma \to G$ such that

$$K_1(y) \cong K_1\big(C^*(y)\big)$$

for all y.

Proof: By Bott periodicity, $K_1C^*(y) \cong K_0\,(C\,(\,(0,1),A))$ and by [262] and Proposition (4.1.8) the right-hand side can be computed by a Borel function.

Theorem (4.1.12)[257]: There is a Borel map Ell:$\Gamma_{\mathrm{u}} \to$ Ell such that Ell (γ) is the Elliott invariant of $C^*(\gamma)$, for all $\gamma \in \Gamma$.

Proof. The computation of K -theory is Borel by Proposition (4.1.8) and Proposition (4.1.11) By Farah et al. [262], the computation of the tracial simplex $\mathbb{T}(y) \cong TC^*(y)$is Borel as well. Since $\phi \in \mathbb{T}(y)$is identified with a continuous map on a dense subset of $C^*(y)$, by restricting this map to Proj (y) and then composing with the embedding of Proj (y) into $K_0(y)$ we obtain the restriction of the coupling map $r_{C^*(y)}$ to the positive part of $K_0(y)$.The coupling map is now canonically extended to $K_0(y)$.

[262] defined an alternative space of Choquet simplexes and showed that it is weakly equivalent to the more straightforward one used above.

We show that the Cuntz semigroup of a separable C^*-algebra is Borel computable, as is a related invariant, the radius of comparison. The relevance of the Cuntz semigroup to C^*-algebra classification was demonstrated in [271], where it was used to distinguish simple unital separable nuclear C^* -algebras with identical Elliott invariants; see also [108]. We review the basic properties of Cuntz semigroups below, see [164].

Let A be a C^*algebra. Define on $(A \otimes \mathcal{K})_+$ a pinary relation by letting $a \lesssim b$ if and only if $v_n b v_n^* \to a$ for some sequence v_n in $A \otimes \mathcal{K}$. Let us write $a{\sim}b$ if $a \lesssim b$ and $b \lesssim a$. In this case, we say that a is Cuntz equivalent to b. Let $Cu\ (A)$ denote the set $(A \otimes \mathcal{K})_+/{\sim}$ of Cuntz equivalence classes. We use $[a]$ to denote the class of a in $Cu(A)$. It is clear that $[a] \leq [b] \Leftrightarrow a \lesssim b$ defines an order on $Cu(A)$. We also endow $Cu\ (A)$ with an addition operation by setting $[a] + [b] := [a' + b']$, where a' and b' are orthogonal and Cuntz equivalent to a and b respectively (the choice of a' and b' does not affect the Cuntz class of their sum).

The semigroup $Cu\ (A)$is an object in a category of ordered Abelian monoids denoted by Cu, a category in which the relation of order-theoretic compact containment plays a significant role, see [258]. Let T be a preordered set with $x, y \in T$. We say that x is compactly contained in y—denoted by $x \ll y$—if for any increasing sequence $y_n$in T with supremum y, we have $x \leq y_{n_0}$ for some $n_0 \in \mathbb{N}$. An object S of Cu enjoys the following properties:

(P1) S contains a zero element;

(P2) the order on S is compatible with addition: $x_1 + x_2 \leq y_1 + y_2$ whenever $x_i \leq y_i, i \in \{1,2\}$;

(P3) every countable upward directed set in S has a supremum;

(P4) the set $x_{\ll} = \{y \in S\ |y \ll x\}$ is nonempty and upward directed with respect to both $\leq$ and $\ll$, and contains a sequence (x_n) such that $x_n \ll x_{n+1}$ for every $n \in \mathbb{N}$ and $\sup_{\mathrm{n}} x_n = x$;

(P5) the operation of passing to the supremum of a countable upward directed set and the relation $\ll$ are compatible with addition: if S_1 and S_2 are countable upward directed sets in S, then $S_1 + S_2$ is upward directed and $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$, and if $x_i \ll \mathrm{y_i}$ for $i \in \{1,2\}$, then $x_1 + x_2 \ll y_1 + y_2$.

Here we assume further that $0 \leq x$ for any $x \in S$. This is always the case for $Cu(A)$.For S and T objects of Cu the map $\phi: S \to T$ is a morphism in the category Cu if

(M1) ϕ preserves the relation $\leq$;

(M2) ϕ is additive and maps 0 to 0;
(M3) ϕ preserves the suprema of increasing sequences;
(M4) ϕ preserves the relation $\ll$.

Definition (4.1.13)[257]: Let $S \in Cu. A$ countable subset D of S is said to be sup-dense if each $s \in S$ is the supremum of a $\ll$-increasing sequence in D. We then say that S is countably determined. (Here by $\ll$ we mean the relation$\ll$ on D inherited from $S, i.e., d_1 \ll d_2$ in D iff $d_1 \ll d_2$ in S.)

Definition (4.1.14)[257]: Let Cu_0 denote the category of pairs (S, D) where S is a countably determined element of Cu, and D is a distinguished sup-dense subset of S which is moreover a semigroup with the binary operation inherited from S. We further assume D to be equipped with the relations $\leq$ and $\ll$ inherited from S.

An element x of $S \in Cu$ such that $x \ll x$ is compactly contained in itself, or briefly compact. If $(S, D) \in Cu_0$ then D automatically contains all compact elements.

Let C be the space of triples $(\oplus, \lesssim, \ll)$ in $\mathbb{N}^{\mathbb{N}\times\mathbb{N}} \times \mathcal{P}(\mathbb{N}\times\mathbb{N}) \times \mathcal{P}(\mathbb{N}\times\mathbb{N})$ with the following properties:

(i) $(\mathbb{N}, \oplus, \lesssim, \ll)$ is an ordered semigroup under the order $\lesssim$ (we will use a, b, c, etc. to represent elements of the semigroup);

(ii) $\ll$ is a transitive antisymmetric relation with the property that $a \ll b$ and $c \ll d$ implies $a \oplus c \ll b \oplus d$;

(iii) $a \ll b$ implies $a \lesssim b$;

(iv) for each a in the semigroup, there is some $b \ll a$, and if a does not satisfy $a \ll a$, then the set of all such b is upward directed and has no maximal element.

(Warning: $\ll$ here is not defined in terms of $\lesssim$ as in our discussion of the Cuntz semigroup, but rather is just some other relation finer than $\leq$. It will coincide with the Cuntz semigroup definition in the case that an element of C really is a sup-dense subsemigroup of an element of the category Cu.) We can define a map $\Phi: Cu_0 \to C$ in an obvious way: send (S, D) to the triple $\oplus, \lesssim, \ll$ corresponding to D ($D = \{d_n : n \in \mathbb{N}\}$ is the ordered semigroup on $\mathbb{N}$ defined by $m \oplus n = k$ if and only if $d_m + d_n = d_k, m \lesssim n$ if and only if $d_m \lesssim d_n$ and $m \ll n$ if and only if $d_m \ll d_n$).

If $D \in C$, we let $D^{\nearrow}$ denote the set of $\ll$-increasing sequences in D. Define an equivalence relation on $\approx$ on $D^{\nearrow}$ by

$$(x_n) \approx (y_n) \Leftrightarrow (\forall m)(\exists n) x_m \ll y_n \text{ and } y_m \ll x_n.$$

Equip $D^{\nearrow}$ with the relations

$$(x_n) \leq^{\nearrow} (y_n) \Leftrightarrow (\forall n)(\exists m) x_n \leq y_m$$

and

$$(x_n) \leq^{\nearrow} (y_m) \Leftrightarrow (\exists m_0)(\forall n) x_n \lesssim y_{m_0}.$$

Note that

$$(x_n) \leq^{\nearrow} (y_n) \wedge (y_n) \leq^{\nearrow} (x_n) \Leftrightarrow (x_n) \approx (y_n).$$

Define $(x_n) \oplus^{\nearrow} (y_n) = (x_n \oplus y_n)$ and set

$$W(D) = D^{\nearrow}/\approx \text{ and } W(S) = S^{\nearrow}/\approx\cdot$$

Note that the operation $\oplus$ and the relations $\leq^{\nearrow}$ and $\ll^{\nearrow}$ drop to an operation $+$ and relations $\lesssim$ and $\ll$ on $W(D)$, respectively.

If (S, D) in Cu_0, then the semigroup D is an element of the category Pre Cu introduced in [100], and S is a completion of D in Cu in the sense of [100]. An appeal to [100] shows that S, too, is a completion of D in Cu, whence

$$W(D) \cong W(S) \cong S$$

In Cu.

Let y be the Borel space of all functions from $\mathbb{N}$ to the Baire space $\mathbb{N}^N$. Since $\alpha \in y$ is a map from $\mathbb{N} \to \mathbb{N}^N$ and the elements of C have $\mathbb{N}$ as the underlyingset, ifD_1 and D_2 in C are fixed then $\alpha \in y$ codes a map from D_1 to $D^{\mathbb{N}}$. We shall identify α with this map whenever D_1 and D_2 are clear from the context. The set of all triples (D_1, D_2, α) such that the range of α is included in $D_2^{\nearrow}$ is a closed subset of $C^2 \times y$. To each $D \in C$, we associate a map $\eta D: D \to D^{\nearrow}$ (or simply η if D is clear from the context) as follows: Select, in a Borel manner, a sequence $\eta D(a) = (a_n)$ which is cofinal in $\{b \in D \mid b \ll a\}$. The association $D \mapsto \eta D$ is then Borel. If $f: W(D_1) \to W(D_2)$ is a semigroup homomorphism preserving $\leq$ and $\ll$ and $\alpha \in y$, then we say that α codes F if $[\alpha(a)]F[\eta(a)]$ for all $a \in D_1$. Note that α really codes the restriction of F to $\eta(D_1)$,but, as we shall see, F is completely determined by this restriction if $W(D_1)$and $W(D_2)$ are in the category Cu.

Lemma (4.1.15)[257]: Let $(S, D) \in Cu_0$. Then $(a_n) \ll^{\nearrow} (b_n)$ in $D^{\nearrow}$ if and only if $[a_n] \ll [b_n]$ in $W(D) \cong S$, where $\ll$ is the relation of order-theoretic compact containment inherited from the relation $\leq$ on S.

Proof: suppose first that$(a_n) \ll^{\nearrow} (b_m)$, and fix m_0 such that $a_n \precsim b_{m_0}$ for all n.We must prove that if, for fixed j, $\left(C_n^j\right) \in D^{\nearrow}$, and if moreover$\left[C_n^j\right]$ is $a \leq$ increasing sequence in j with supremum $[b_n]$, then $[(a_n)] \leq \left[C_n^{j_0}\right]$ for some $j_0 \in \mathbb{N}$. first we recall (see the proof of the existence of suprema in inductive limits of cuntz semigroups in [258]) that for such $\left(C_n^j\right)$, there is a sepuence of natural numbers (n_j) with the property that $\left(C_n^j\right) \approx (b_m)$. In particular,there is j_0 such that $b \ll C_{n_{j_0}}^{j_0}$. Since $\left(C_k^{j_0}\right) \in D^{\nearrow}$, we have

$$\left(C_n^j\right) \approx (b_n)\, a_n \ll b_{m_0} \ll C_{n_{j_0}}^{j_0}$$

for all $n \in \mathbb{N}$, and so $(a_n) \ll^{\nearrow} \left(C_k^{j_0}\right)$. This implies$[(a_n)] \leq \left[C_k^{j_0}\right]$as required.

For the converse, assume that $[a_n] \ll [b_m]$.Since $b_m \ll b_{m+1}$, we know that for any element of the sequence $\eta(b_m)$, there is an element of the sequence $\eta(b_{m+1})$ that $\ll$-dominates it, so that $[\eta(b_m)] \leq [(b_{m+1})]$ There is also, for given m, and element of the sequence $\eta(b_{m+1})$that $\ll$-dominates b_m. (These two assertions follow from property (4) in the definition of C.) It follows that for some sequence m_j, we have $[\eta(b_j)_{m_j}] \geq [(b_m)]$.

Identifying $\eta(b_j)$ with $\left(C_n^j\right)$from the first part of the proof and observing (see again the proof of existence of suprema in inductive limits of Cuntz semigroups in [258]) that the n_j chosen above can be increased without disturbing the fact $\left(C_n^j\right) \approx (b_n)$we see that by increasing the $m_j$if necessary, we also have that $\sup_j[\eta(b_j)] = [\eta(b_j)_{m_j}] \geq [(b_m)]$. It follows that $\left[\eta(b_{j_0})\right] \geq [(a_n)]$ for some j_0, whence $a_n \ll b_{j_0}$ for all n, as required.

Lemma (4.1.16)[257]: Let $(S, D) \in Cu_0$. Then $a \ll b$ in D if and only if $[\eta(a)] \ll [\eta(b) \in W(D) \cong S$.

Proof: By Lemma (4.1.15), it will suffice to prove that $a \ll b$ iff $\eta\ (a) \ll^{\nearrow} \eta\ (b)$ in $D^{\nearrow}$. Suppose first that $a \ll b$, so that $(\eta\ (a)_i)$ and $(\eta\ (b)_m)$are $\ll$-increasing sequences in D with suprema a and b, respectively (this uses several acts: that D is embedded in $S \in Cu$;that objects in Cu admit suprema for increasing sequences; and that S may be identified with $W(S)$). Since $a \ll b$,there is $m_0$such that $\eta\ (b)_m \geq a \geq \eta\ (a)_i$ for all i and for all $m \geq m_0$, as required.

Conversely, suppose that $\eta(a) \ll^{\nearrow} \eta(b)$, so that there is m_0 such that $\eta\ (a)_i \leq \eta\ (b)_m$ for all $m \geq m_0$. Now

$$\sup_i \eta\ (a)_i = a \leq \eta\ (b)_m \ll \eta\ (b)_{m+1} \leq b.$$

so that $a \ll b$.

Lemma (4.1.17)[257]: Let $(S, D) \in Cu_0$. Then $(a_n) \ll^{\nearrow} (b_n)$ in $D^{\nearrow}$ if and only if $[a_n] \leq [b_n]$ in $W(D) \cong S$.

Proof: Suppose that $(a_n) \ll^{\nearrow} (b_n)$. It follows that for each $n \in \mathbb{N}$ there is $m(n)$such that

$$a_n b_{m(n)} \ll b_{m(n)+1}.$$

The statement $[(a_n)] \leq [(b_n)]$ amounts to the existence of $(c_n) \in D^{\nearrow}$ such hat $(a_n) \approx (c_n)$ and $(c_n) \ll^{\nearrow} (b_n)$. Here we can take $(c_n) = (a_n)$, completing the forward implication.

Suppose, conversely, that $[(a_n)] \leq [(b_n)]$, so that there is some $(c_n) \in D^{\nearrow}$ such that $(a_n) \cong (c_n)$and $(c_n) \leq (b_n)$.Since $(a_n)$and $(c_n)$are cofinal in each other with respect to $\ll$, it is immediate that $(a_n) \leq^{\nearrow} (b_n)$.

Lemma (4.1.18)[257]: Let $(S, D) \in Cu_0$.Then $a \leq b$ in D if and only if $[\eta(a)] \leq [\eta(b)]$ in $W(D) \cong S$.

Proof: By Lemma (4.1.17), it is enough to prove that $a \leq b$ iff $\eta(a) \leq^{\nearrow} \eta(b)$ in $D^{\nearrow}$.Suppose first that. $a \leq b$. The sequence $(\eta(a)_n)$,being cofinal with respect to $\ll$ in $\{c \in D \mid c \ll a\}$, has a supremum in S, namely, a itself. A similar statement holds for b. For any $n \in \mathbb{N}$, we have $\eta(a)_n \ll a$, and $\sup \eta(b)_m = b \geq a$. It follows that $\eta(b)_m \gg \eta(a)_n$ for all m sufficiently large, whence $[\eta(a)] \leq [\eta(b)]$, as desired.

Suppose, conversely, that $\eta(a) \leq^{\nearrow} \eta(b)$ in $D^{\nearrow}$. Since $\sup \eta(a)_n = a, \sup \eta(b)_m = b$, and for reach n there is m such that $\eta(a)_n \ll \eta(b)_m$, it is immediate that $a \leq b$ in S.

Using methods similar to those of Lemmas (4.1.15)– (4.1.18) one can also prove the following result.

Lemma (4.1.19)[257]: Let $(S, D) \in Cu_0$. Then the following are equivalent:

$$a \oplus b = c \text{ in } D;$$

$$\eta(a) \oplus^{\nearrow} \eta(b) = \eta\ (c)] \text{in } D^{\nearrow};$$

(iii) $[\eta(a)] + [\eta(b)] = [\eta\ (c)]$ in $W\ (D)$.

Lemma (4.1.20)[257]: Let $D_1, D_2, \in C$ be sup-dense subsemigroups of elements of Cu. If α codes a homomorphism $\Phi: W(D_1) \to W(D_2)$, then for $a, b \in D_1$ we have:

$a \lesssim b$ implies $(\forall m)(\exists n)\alpha(a)_m \lesssim \alpha(b)_n$;

$a \ll b$ implies $(\exists n)(\forall m)\alpha(a)_m \lesssim \alpha(b)_n$;

$\alpha(a) \oplus \alpha(b)$ (defined pointwise) satisfies $\alpha(a) \oplus \alpha(b) \approx \alpha(a \oplus b)$.

Conversely, if α has properties (i)‑ (iii), then

$$(\psi: \eta(D_1) /\approx) \cong D_1 \to W(D_2)$$

To see that $\leq$ is preserved by Φ on $W(D_1)$, consider $[(b_n)] \leq [(c_m)]$. Passing to subsequences we can assume that $b_k \ll c_k$ for every k. Then by property (ii) and Lemmas (4.1.15) and (4.1.16) we have

$$[\alpha(b_k)] \ll [\alpha(c_k)] \Rightarrow \sup_k[\alpha(b_k)] \leq \sup_k[\alpha(c_k)] \Rightarrow \psi[(b_n)] \leq \psi[(c_m)].$$

We shall now define an analytic equivalence relation on C which, for sup-dense subsemigroups of elements of Cu, amounts to isomorphism. Consider the standard Borel space $C^2 \times y^2$. In this space consider the Borel set χ consisting of all quadruples $(D_1, D_2, \alpha_1, \alpha_2)$ such that

(i) α_1 and α_2 satisfy the (Borel) conditions (i)- (iii) of Lemma (4.1.17);

(ii) $(\forall a \in D_1)(\forall b \in D_2)$ we have

$$\alpha_1(a) \ll^{\nearrow} \eta(b) \Leftrightarrow \eta(a) \ll^{\nearrow} \alpha_2(b)$$

and

$$\eta(b) \ll^{\nearrow} \alpha_1(a) \Leftrightarrow \alpha_2(b) \ll^{\nearrow} \eta(a).$$

It is straightforward to verify that the conditions above define a Borel subset of $C^2 \times y^2$, whence χ is a standard Borel space. Now define a relation E on C by

$$D_1 E\, D_2 \Leftrightarrow (\exists \alpha_1, \alpha_2)\ (D_1, D_2, \alpha_1, \alpha_2) \epsilon \chi$$

whence E, as the co-ordinate pro jection of χ onto C^2, is analytic.

Proposition (4.1.21)[257]: Let $D_1, D_2 \in C$ be sup-dense subsemigroups of elements of Cu. It follows that $D_1 E\, D_2$iff $W(D_1) \cong W(D_2)$in the category Cu.

Proof: Assume $W(D_1) \cong W(D_2)$and let $\phi: W(D_1) \to W(D_2)$ be an isomorphism. Pick α_1 that codes ϕ and α_2 that codes ϕ^{-1}, so that α_1 and α_2 have the properties (i)- (iii) of Lemma (4.1.20). For (ii) in the definition of X, we will only prove the first equivalence, as the second one is similar. By Lemma (4.1.15), the first equivalence in (i) is equivalent to

$$(\forall a \in D_1)(\forall b \in D_2)[\alpha_1(a)] \ll [\eta(b)] \Leftrightarrow [\eta(a)] \ll [\alpha_2(b)] \cdot$$

Suppose $[\alpha_1(a)] \ll [\eta(b)]$, so that

$$\phi^{-1}[\alpha_1(a)] \ll \phi^{-1}[\eta(b)]$$

(morphisms in Cu preserve $\ll$). Since α_2 codes ϕ^{-1},the right hand side above can be identified with $[\alpha_2(b)]$. Similarly, $\phi^{-1}[\alpha_1(a)] = \phi^{-1}\phi[\eta(a)] = [\eta(a)]$, so that $[\,\eta(a)] \ll [\alpha_2(b)]$. The other direction is similar, establishing (ii) from the definition of X, whence $D_1 E\, D_2$.

Now assume $(D_1, D_2, \alpha_1, \alpha_2)\epsilon\chi$ for some α_1 and α_2. Using Lemma (4.1.20) we obtain homomorphisms $\phi_1: W(D_1) \to W(D_2)$and $\phi_2: W(D_1) \to W(D_2)$. Let us verify that $\phi_2 \circ \phi_1 = \mathrm{id}W(D_1)$ (the proof for that $\phi_1 \circ \phi_2 = \mathrm{id}W(D_2)$ is similar). Fix $[\,(\mathfrak{f}_n)] \in W(D_1)$. Since $a \mapsto [\,\eta(a)]$ is a complete order embedding of D_1 into $W(D_1)$ relative to$\lesssim$ and$\ll$ by Lemma (4.1.15), we have $[(\mathfrak{f}_n\,)] = \sup_n[\eta\,(\mathfrak{f}_n)]$. Since ϕ_1 preserves $\ll$, we have a corresponding $\ll$ -increasing sequence $\phi_1[\eta(\mathfrak{f}_n)] = [\alpha(\mathfrak{f}_n\,)], i \in \mathbb{N}$ (see Lemma (4.1.20)). Choose a $\ll$- increasing sequence $[\eta(b_i)]$ in $W(D_2)$ with supremum $\phi_1[(\mathfrak{f}_n\,)]$, and note that this is also the supremum of the sequence $[\alpha_1(\mathfrak{f}_n\,)]$. Since $W(D_2) \in Cu$ we may, passing to asubsequence if necessary, assume that

$$[\eta(b_i)] \ll [\alpha_1(\mathfrak{f}_i)]\text{and}[\alpha_1(\mathfrak{f}_i)] \ll [\eta(b_{i+1})] \cdot$$

Using (ii) in the definition of X and the relations above we obtain

$$[\alpha_2(b_i)] \ll [\eta(\mathfrak{f}_i)]\text{and}[\eta(\mathfrak{f}_i)] \ll [\alpha_2(b_{i+1})],$$

so that the sequences $[\alpha_2(b_i)]$and $[\eta(\mathfrak{f}_i)]$ have the same supremum, namely, $[\,(\mathfrak{f}_i)]$. Now we compute:

$$(\phi_2 \circ \phi_1)[(\mathfrak{f}_n\,)] = (\phi_2 \circ \phi_1) \sup_i[\eta(\mathfrak{f}_i\,)]$$

$$= \phi_2 \left(\sup_i \phi_1[\eta(\mathfrak{f}_i)]\right)$$
$$= \phi_2 \left(\sup_i [\alpha_1(\mathfrak{f}_i)]\right)$$
$$= \phi_2 \left(\sup_i [\eta(\mathfrak{f}_i)]\right)$$
$$= \left(\sup_i \phi_2[\eta(\mathfrak{f}_i)]\right)$$
$$= \left(\sup_i [\alpha_2(\mathfrak{f}_i)]\right)$$
$$= \left(\sup_i [\eta(\mathfrak{f}_i)]\right)$$
$$= [(\mathfrak{f}_n)].$$

Recall the following well-known lemma.

Lemma (4.1.22)[257]: For any strictly decreasing sequence ϵ_n of positive tolerances converging to zero, the sequence $\langle(a-\epsilon_n)_+\rangle$ $\ll$-increasing in $Cu(A)$.

In some cases, for example when a is a pro jection, the sequence in the Lemma is eventually constant, that is, $\langle a\rangle$ is compact. This occurs, for instance, when $a \precsim (a-\epsilon)_+$ for some $\epsilon > 0$.

Proposition (4.1.23)[257]: There is a Borel map $\psi: \Gamma \to C$ such that $W(\psi(y)) \cong Cu(C^*(y))$.

Proof: Fix $y_0 \in \Gamma$ such that $C^*(y_0)$ is the algebra of compact operators and a bijection π between $\mathbb{N}^2$ and $\mathbb{N}$. Moreover, choose y_0 so that all operators in y_0 have finite rank and y_0 is closed under finite permutations of a fixed basis (e_n) of H. We also fix a sequence of compact partial isometries v_m, such that v_m swaps the first m vectors of (e_n) with the next m vectors of this basis. This sequence will be used in the proof of Claim (4.1.25).

Let ψ denote the Borel map from Γ to Γ obtained as the composition of three Borel maps: Tensor $(\cdot, y_0)$, where Tensor is the Borel map from [262]; the map $y \mapsto (a_n(y))$ (see [262]); and finally the map that sends (a_n) to (b_n) where

$$b_n = \left(\left(a_{\pi_0(n)}a^*_{\pi_0(n)}\right) - 1/\pi_1(n)\right)_+ \quad (1)$$

(here $n \mapsto (\pi_0(n), \pi_1(n))$ is the fixed bijection between $\mathbb{N}$ and $\mathbb{N}^2$).

Fix $y \in \Gamma$. Then y_1 Tensor (y, y_0) satisfies $C^*(y)\otimes\mathcal{K} \cong C^*$ Tensor (y, y_0) Moreover, for any two positive entries a and b if y_1 there are orthogonal positive a' and b' in y_1 such that $a \sim a'$ and $b \sim b'$. (Here $\sim$ denotes Cuntz equivalence.) If $y_2 = (p_n(y_1))$, then the elements of y_2 are norm-dense in $C^*(y_2) \cong C^*(y)\otimes\mathcal{K}$, and y_2 contains y_1 as a subsequence. Finally, if y_3 is the sequence as in (1), then y_3 is a norm-dense set subset of the positive elements of $C^*(y)\otimes\mathcal{K}$. Let us write $d_m(y) := (y_3)m)$ and $x_n(y) := (y_2)n$.

Claim (4.1.24)[257]: The map $\Gamma \to P\,(\mathbb{N})^2: \gamma \overset{\psi_{\precsim}}{\mapsto} R[\precsim, y]$, defined by

$$(m, n) \in R[\precsim, y] \text{if and only if } d_m\,(y) \precsim d_n\,(y)$$

(where $\precsim$ is computed in $C^*(y_1) = C^*(y)\otimes\mathcal{K}$ is Borel.

Proof: Recall that a map is Borel if and only if its graph is Borel. We have $\left(\text{writing } d_m \text{ for } d_m(y) \text{ and } x_m \text{ for } x_m(y)\right)$ $(m, n) \in R[\precsim, y]$ if and only if

$$(\forall_i)(\exists_j)\|x_j d_n x_j^* - d_m\| < 1/i,$$

Therefore, the graph of $\psi_{\lesssim}$ is equal to $\bigcap_i \bigcup_j A_{ij}$ where all of these sets are Borel since the maps $y \mapsto d_m(y)$ and $y \mapsto d_m(y)$ are, by the above, Borel. The map that sends the pair of sequences $x_j$and d_j to $R[\lesssim, y]$is therefore Borel.The computation of these two sequences from y is Borel by construction, and this completes the proof. $\Gamma \to \mathrm{P}(\mathbb{N})^3: y \xrightarrow{\psi_+} R[+, y]$,
By Claim (4.1.24), for each y we have a preordering $R[\lesssim, y]$on $\mathbb{N}$. Then

$$R[\sim, y] = \{(m,n): (m,n) \in R[\lesssim, y] \text{ and } (n,m) \in R[\lesssim, y]\}$$

is also a Borel function, and it defines a quotient partial ordering on $\mathbb{N}$ for every y. In what follows we use $[a]$ to denote the Cuntz equivalence class of a positive element of $C^*(y)$.

Claim (4.1.25)[257]: the map / $\Gamma P(\mathbb{N})^3: y \xrightarrow{\psi_+} R[+, y]$, defined by

$$(m,n,k) \in R[+, y] \text{if and only if } [d_m] + [d_n] = [d_k]$$

(where $+$ is computed in $(Cu(C^*(y))))$ is Borel. Moreover, it naturally defines a semigroup operation on $\mathbb{N}/R[\sim, y\ ]$.

Proof: Fix y. Let us first prove that the sequence $d_m := d_m(y)$ is such that for all m and n there is k satisfying $[(d_m)\ ] + [d_n\ ] = [d_k\ ]$. Our choice of generating sequence y_0 for K ensures that each d_m is contained in $C^*(y) \otimes M_n$ for some n, where $M_1 \subseteq\ M_2 \subseteq M_3\ \cdots$ is a fixed sequence of matrix algebras with union dense in K. The M_n are the bounded operators on span $(e_1, \ldots, e_n)$. By construction y_0 is closed under finite permutations of the basis (e_n), so that for a large enough l the isometry v_l (see above) we have that $d_m := (1 \otimes v_l) d_m (1 \otimes v_l) *$ is both Cuntz equivalent to d_m and orthogonal to d_n. Here the "1" in the first tensor factor is the unit of $C^*(y)$
if $C^*(y)$ is unital, and the unit of the unitization of $C^*(y)$ otherwise. Note that $\omega_1 := d_m(1 \otimes v_l)$belongs to $C^*(y) \otimes \mathcal{K}$ and that $\omega_l d_m \omega_l^* = d_m^3$ is cuntz equivalence to and orthogonal to d_m. It follows that

$$[d_n + \omega_l d_m \omega_l^*] = [d_n] + [\omega_l d_m \omega_l^*] = [d_n] + [d_m].$$

By the defininition of the function Tensor in [262], for all l we have $\omega_l d_m \omega_l^* = d_{r(l)}$and $d_n + d = d_{k(l)}$ for some $r(l)$ and $k(l)$.

Now we check that the graph of ψ_+ is Borel. This is equivalent to verifying that the graph of the function that maps each triple (y, m, n) to the set $X_{y,m,n}$, of all k such that $(m, n, k) \in \psi_+(y)$ is Borel. Moreover, a function Λ from a Borel space int $\mathcal{P}(\mathbb{N})$is Borel if and only if all of the sets $\{(\gamma, k): k \in \Lambda\ (\gamma)\}$ are Borel.

It will therefore suffice to check that the set $\{\ (\gamma, (m, n, k)): (m, n, k) \in\ \psi_+\ (\gamma)\}$ is Borel. But by the above, $(m, n, k) \in \psi_+\ (\gamma)$ is equivalent to (writing d_m for $d_m(y)$)

$$(\exists m)(\forall l \geq m) d_n + \omega_l d_m \omega_l^* \sim d_k.$$

where $\sim$ is the Cuntz equivalence relation: $a \sim b$ iff $a \lesssim b$ and $b \lesssim a$. This is a Borel set, and therefore the map ψ_+ is Borel.

Clearly, $\psi_+(y)$ is compatible with $\lesssim$and it defines the addition on $\mathbb{N}/R[\sim, y]$ that coincides with the addition on the Cuntz semigroup.

Claim (4.1.26)[257]: The map $\Gamma \to P\ (\mathbb{N})^2\ \gamma\ \mapsto\ R[\ll, \gamma]$, defined by

$$(m,n) \in R[\ll, y] \text{if and only if } [d_m] \ll [d_n]$$

(where $\ll$ is computed in $Cu(\ C^*(y))$is Borel.

Proof: We have $[d_m] \ll [d_n]$ if and only if there exists $j \in \mathbb{N}$ such that $d_m \lesssim (d_n - 1/j)_+$ [164].

Recalling that $d_{\pi(n,j)} = (d_n - 1/j)_+$ for all n and j, we see that is equivalent to

$$(\exists j)\big(m,\pi(n,j)\big) \in R[\lesssim, y]$$

and therefore the map is Borel.

Collecting these three claims we see that the map which sends y to an element of C representing $Cu\big(\ C^*(y)\big)$—call it Φ—is Borel.

The radius of comparison is a notion of dimension for noncommutative spaces which is useful for distinguishing simple nuclear C^*-algebras and is connected deeply to Elliott's classification program (see [259], [108]).

Consider the standard space $C_u = C \times \mathbb{N}$, where the second co-ordinate of $(D,e) \in C_u$ represents a distinguished element of D. Let Cu_u denote the category of Cuntz semigroups with a distinguished compact element. It is straightforward, by following the proof of Proposition (4.1.23), to verify that there is a Borel map $\psi\colon \Gamma_u \to C_u$ such $\psi(y) = \big(D,\big[1_{C^*(y)}\big]\big)$, where D is (identified with) a countable sup-dense subsemigroup of $Cu\big(C^*(y)\big)$.

If $(S,e) \in\ Cu_u$, then the radius of comparison of S (relative to e), denoted by $r\ (S,e)$, is defined by $r\ (S,e) = \inf\{m/n \mid m,n \in \mathbb{N} \wedge x \leq y \text{ in } S \text{whenever } (n+1)x + me \leq ny\}$ if this infimum exists, and by $r(S,e) = \infty$ otherwise. Of course, this definition makes sense for any ordered semigroup with a distinguished element e, for example an element (D,e) of $C \times \mathbb{N}$, so we can equally well define $r\ (D,e)$ in the same way.

Proposition (4.1.27)[257]: Let $(\mathrm{S},\mathrm{e}) \in Cu_u$, and let $D \subseteq S$ be a countable sup-subsemigroup of S containing e. It follows that, with respect to the common element $e, r(S,e) = r\ (D,e)$.

Proof: We suppress the e and write only $r(D)$and $r(S)$. It is clear that $r(D) \leq r(S)$. Given $\epsilon > 0$, we will prove $r(S) \leq rD + \epsilon$. Choose $m,n \in \mathbb{N}$ to satisfy

$$r(D) < m/n < r(D) + \epsilon$$

Let $x,y \in S$ satisfy

$$(n+1) \times + me\ \leq ny.$$

There are rapidly increasing sequences (x_k) and (y_k) in D having suprema x and y, respectively. Since e is compact, so is me, that is $me \ll me$. Since $(n+1)x_k \ll (n+1)$ for any k, we can use the fact that addition respects $\ll$ to conclude that

$$(n+1)x_k + me \ll (n+1)x + me \leq ny.$$

It follows that

$$(n+1)x_k + me \ll ny.$$

Since the operation of addition respects the operation of taking suprema, we have sup $ny_l = ny$, whence for some (and hence all larger) $l_k \in \mathbb{N}$, we have

$$(n+1)x_k + me \leq ny_{lk.}.$$

Now since $m\,/\,n > r(D)$ we conclude that $X_k \leq Y_{\iota k}$. Taking suprema yields $X \leq Y$, proving that $r(S) \leq m/n > r(D) + \epsilon$, as desired.

Proposition (4.1.28)[257]: The map $rc\colon \Gamma_u \to \mathbb{R}^+ \cup \{\infty\}$ given by $rc(y) = r\big(Cu\big(C^*(y)\big),\big[I_{C^*(y)}\big]\big)$ is Borel.

Proof: The map $\psi\colon \Gamma_u \to C_u$ is Borel and satisfies $r\big(\psi(y)\big) =\ r\ (Cu\ (C^*\ (y)), [1_{C^*(y)}\ ])$ by Proposition (4.1.27). It will therefore suffice to prove that $r\colon C_u \to \mathbb{R} \cup \{\infty\}$ is Borel. For $m,n \in \mathbb{N}$ the set

$$\mathrm{A}_{m,n}\{(D,e) \in C_u | (\forall x,y \in D)(n+1)x + \mathrm{me} \leq ny \Rightarrow x \leq y\}$$

is Borel. Define a map $\xi_{m,n}: C_u \to \mathbb{R}^+ \cup \{\infty\}$ by declaring that $\xi_{m,n}(D,e) = m/n$ if $(D,e) \epsilon A_{m,n}$ and $\xi_{m,n}(D,e) = \infty$ otherwise. Viewing the $\xi_{m,n}$ as co-ordinates we obtain a Borel map $\xi: C_u \to (\mathbb{R}^+ \cup \{\infty\})^{\mathbb{N}^2}$ in the obvious way, and $(D,e) = \inf \xi(D,e)$.This shows that r is Borel, as desired.

Unlike other invariants of C^*-algebras treated, the theory Th (A) of a C^*-algebra A comes from logic. By the metric version of the Keisler–Shelah theorem [256], it has the property that two C^*-algebras have isomorphic ultrapowers ltrafilters on uncountable sets, even if the algebras in question are separable. A comprehensive tr if and only if they have the same theory. It should be emphasized that the ultrapowers may have to be associated with ultrafilters on uncountable sets, even if the algebras in question are separable. A comprehensive treatment of model theory of bounded metric structures is given in [256], and model-theoretic study of C^*-algebras and tracial von Neumann algebras was initiated in [260], [261].

We now give a special case of the definition of a formula ([261]) in the case of C^*-algebras (cf. [261]). A term is a $*_-$polynomial. A basic formula is an expression of the form $\|p(x_0,\dots,x_{n-1})\|$ where $p(x_0,\dots,x_{n-1})$ is a term in variables $x_0,\dots,x_{n-1}$. Formulas are elements of the smallest set $\mathbb{F}$ that contains all basic formulas and has the following closure properties (we suppress free variables in order to increase readability).

(F1) If $f: \mathbb{R}^n \to \mathbb{R}$ is continuous and $\varphi_1,\dots,\varphi_n$ are formulas, then $f(\varphi_1,\dots,\varphi_n)$ is a formula.

(F2) If φ is a formula, $K \geq \mathbb{N}$ is a natural number, and x is a variable then both $\sup_{\|x\|\leq K} \varphi$ and $\inf_{\|x\|\leq K} \varphi$ are formulas.

Equivalently, formulas are obtained from basic formulas by finite application of the above two operations.

The quantifiers in this logic are $\sup_{\|x\|\leq 1}$ and $\inf_{\|x\|\leq 1}$ A variable appearing in a formula ϕ outside of the scope of its quantifiers (i.e., any φ as in (F2)) is free.

As customary in logic we list all free variables occurring in a fornula φ and write $\varphi(x_0,\dots,x_{n-1})$. A formula $\varphi(x_0,\dots,x_{n-1})$. is interpreted in a C^* -algebra A in a natural way. Given $a_0,\dots,a_{n-1}$ in A, one defines the value $\varphi(a_0,\dots,a_{n-1})^A$ recursively on the complexity of formula φ. As $a_0,\dots,a_{n-1}$ vary, one obtains a function from A^n into $\mathbb{R}$ whose restriction to any bounded ball of A is uniformly continuous ([261]). A sentence is a formula with no free variables. If φ is a sentence then the interpretation φ^A is a constant function and we identify it with the corresponding real number. Theory of a C^*-algebra A is the map $\varphi \longmapsto \varphi^A$ from the set of all sentences int $\mathbb{R}$.

The above definition results in an uncountable set of formulas. However, by restricting terms to $*_-$ polynomials with complex rational coefficients and continuous functions f in (F1) to polynomials with rational coefficients, one obtains a countable set of formulas that approximate every other formula arbitrarily well. Let $\mathbb{S}_0$ denote the set of all sentences in this countable set. Clearly, the restriction of Th (A) to $\mathbb{S}_0$ determines Th (A) and we can therefore consider a closed subset of $\mathbb{R}^{\mathbb{S}_0}$ to be a Borel space of all theories of C^*-algebras.

Proposition (4.1.29)[257]: The function from $\hat{\Gamma}$ into $\mathbb{R}^{\mathbb{S}_0}$ that associates $\mathrm{Th}(C^*(y))$ to $y \in \hat{\Gamma}$ is Borel.

Lemma (4.1.30)[257]: Given a formula $\varphi(x_0,\ldots,x_{n-1})$, the map that associates $\varphi\big(y_{k(0)},\ldots,y_{k(n-1)}\big)^{C^*(y)}$ to a pair $\big(y,\vec{k}\big) \in \hat{\Gamma} \times \mathbb{N}^n$ is Borel.

Proof: By recursion on the complexity of φ. We suppress parameters $\psi(y)x_0,\ldots,x_{n-1}$ for simplicity. If φ is basic, then the lemma reduces to the fact that evaluation of the norm of a $*$-polynomial is Borel-measurable. The case when φ is of the form $f\ (\varphi_0,\ldots,\varphi_{n-1})$ as in (F1) and lemma is true for each φ_i is trivial.

Now assume φ is of the form $\sup_{\|x\|\le K}\psi(y)$ with $K \ge 1$. Function $t_k\colon \mathbb{R} \to \mathbb{R}$ defined by $t(r) = r,$ if $r \le K$ and $t(r) = 1/r$ if $r > K$ is continuous, and since

$$\varphi^{C^*(y)} = \sup_{i\in\mathbb{N}} \sup_{i\in\mathbb{N}} \psi(k(\|y_i\|)y_i),$$

we conclude that the computation of φ is Borel as a supremum of countably many Borel functions. The case when φ is in $\inf_{\|y\|\le k}\psi(y)$ is similar.

We note that an analogous proof shows that the computation of a theory of a tracial von Neumann algebra is a Borel function from the corresponding subspace of Effros–Mar´echal space into $\mathbb{R}^{\mathbb{S}_0}$.

The stable rank sr (A) of a unital C^*-algebra A is the least natural number n such that

$$L_{g_n} = \left\{(a_1,\ldots,a_n) \in A^n \mid \exists b_1,\ldots,b_n \in A \text{such that} \left\|\sum_{i=1}^{n} b_i a_i - 1_A\right\| < 1\right\}$$

is dense in A^n, if such exists, and ∞ otherwise. The real rank $rr\ (A)$ is the least natural number n such that

$$Lg_{n+1}^{sa} = \left\{(a_1,\ldots,a_{n+1}) \in A_{sa}^{n+1} \mid \exists b_1,\ldots,b_{n+1} \in A_{sa} \text{such that} \left\|\sum_{i=1}^{n} b_i a_i - 1_A\right\| < 1\right\}$$

where A_{sa} denotes the self-adjoint elements of A. Again, if no such n exists, we say that $rr(A) = \infty$.

Theorem (4.1.31)[257]: The maps $SR\colon \Gamma \to \mathbb{N} \cup \{\infty\}$ and $RR\colon \Gamma \to \mathbb{N} \cup \{\infty\}$ given by $SR(y) = sr\big(C^*(y)\big)$ and $RR(y) = rr\big(C^*(y)\big)$, respectively, are Borel.

Proof: We treat only the case of $SR\ (\bullet)$; the case of $RR\ (\bullet)$ is similar. We have

$$C^*(y) \in L_{g_n} \Leftrightarrow \big(\forall_{i_1} < i_2 < \cdots i_n\big)\big(\exists j_1 < j_2 < \cdots < j_{n,}\big)\colon \left\|\sum\ \right\|$$

For fixed $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_{n,}$ the set on the left hand side is norm open in all co-ordinates $\mathcal{B}(\mathcal{H})^{\mathbb{N}} = \Gamma$, and hence Borel. The theorem follows immediately.

The Jiang-Su algebra Z plays a central role in the classification theory of nuclear separable C^*algebras. Briefly, one can expect good classification results for algebras which are Z-stable, i.e., which satisfy $A \otimes Z \cong A$ (see [259] for a full discussion). We prove here that the subset of Γ consisting of Z-stable algebras is Borel.

It was shown in [264] that Z can be written as the limit of a C^*-algebra inductive sequence

$$Z_{n_1,n_1+1} \xrightarrow{\phi_1} Z_{n_2,n_2+1} \xrightarrow{\phi_2} Z_{n_3,n_3+1} \xrightarrow{\phi_3} \cdots,$$

Where

$$Z_{n,n+1} = \{f \in C([0,1]; M_n \otimes M_{n+1}) \mid f(0) \in M_n \otimes 1_{n+1}, f(1) \in 1_n \otimes M_{n+1}\}$$

is the prime dimension drop algebra associated to n and $n+1$. The property of being Z-stable for a C^*-algebra A can be characterized as the existence, for each n, of a sequence of $*$_homomorphisms $\psi_k: Z_{n,n+1} \rightarrow A$ with the property that

$$\|[\psi_k(f), x]\| \rightarrow 0, \forall f \in Z_{n,n+1}, \forall x \in A.$$

The algebra $Z_{n,n+1}$ was shown in [264] to admit weakly stable relations, i.e., there exists a finite set of relations $\mathcal{R}_n$in $l\,(n)$ indeterminates with the following properties:

(i) the universal C^*-algebra for $\mathcal{R}_n$is $Z_{n+1;}$

(ii) for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $g_1, \ldots, g_{l\,(n)}$ are elements in a C^*-algebra A which satisfy the relations $\mathcal{R}_n$ to within $\delta\,(\epsilon)$, then there exist $h_1, \ldots, h_{l\,(n)} \in A$ which satisfy the relations $\mathcal{R}_n$ precisely and for which $\|g_i - h_i\| < \epsilon$.

What is really relevant for us is that if $g_1, \ldots, g_{l\,(n)}$ are elements in a C^*-algebra A which satisfy the relations $\mathcal{R}_n$ to within $\delta\,(\epsilon)$, then there is a $*$_homomorphism $\eta: Z_{n,n+1} \rightarrow A$ such that the indeterminates for $\mathcal{R}_n$ are sent to elements ϵ-close to $g_1, \ldots, g_{l(n)}$, respectively.

Using the equivalence of the parameterizations Γ and $\hat{\Gamma}$ for separable C^*-algebras, we may assume that the sequence y in $\mathcal{B}(\mathcal{H})^{\mathbb{N}}$ giving rise to $C^*(y)$ is in fact dense in $C^*(y)$.The Z-stability of $C^*(y)$ for $y = (a_i)_{i\in\mathbb{N}}$ is then equivalent to the following statement:

$(\forall k)(\forall n)(\forall j)\left(\exists\left(i_1, \ldots, i_{l(n)}\right)\right)$such that $a_{i1}, \ldots, a_{il(n)}$are a $\delta\,(1/k)$representation of $\mathcal{R}_n$ and$\|\,[a_{is}, a_m]\| < 1/k$ for each $s \in \{1, \ldots, l\,(n)\}$ and $m \in \{1, \ldots, j\}$.

If we fix k, n, j and $(i_1, \ldots, i_{l(n)})$it is clear that those $y \in \hat{\Gamma}$ for which $(a_{i1}, \ldots, a_{il(n)})$ satisfy the latter two conditions above form a norm open and hence Borel set. This theorem follows immediately:

Theorem (4.1.32)[257]: $\{y \in \Gamma \mid C^*\,(y)$ is Z- stable$\}$ is Borel.

A completely positive map $\emptyset: A \rightarrow B$ between C^*-algebras has order zero if it is orthogonality preserving, in the sense that for positive a, b in A we have $ab = 0$ implies $\phi(a)\phi(b) = 0$

A C^*-algebra A has nuclear dimension at most n if the following holds. For every $\varepsilon > 0$, for every finite $F \subseteq A$, there are finite-dimensional C^*-algebras $B_1, \ldots, B_n$ and completely positive maps $\psi: A \rightarrow \bigoplus_{i=1}^{n}$ $\psi: A \rightarrow \bigoplus$ such that

(i) $\|\psi \circ \phi\,(a) - a\| < \varepsilon$ for all $a \in F$,

(ii) $\|\psi\| \leq 1$, and

(iii) $\phi \restriction Bi$ has order zero for every $i \leq n$.

The nuclear dimension of A, denoted dimnuc A,is the minimal n (possibly ∞) such that A has nuclear dimension $\leq n$ (see [196]).

The proof of the following theorem is based on Effros's proof that nuclear C^*-algebras form a Borel subset of Γ (see [265]).

Theorem (4.1.33)[257]: The map $\dim_{\text{nuc}}: \Gamma \rightarrow \mathbb{N} \cup \{\infty\}$ is Borel.

Proof: It suffices to check that the set of all y such that $\dim_{\text{nuc}}\left(C^*(y)\right) \leq n$ is Borel. Let $M_n(A^*)$denote the space of $n \times n$ matrices of the elements of the Banach space dual of A, naturally identified with the space of bounded linear maps from A into $M_n(\mathbb{C})$.We consider this space with respect to the weak* topology, which makes it into a K_σ Polish space.

As demonstrated in [265], there is a Borel map $\Upsilon: \Gamma \times n \to \left(M_n(A^*)\right)^{\mathbb{N}}$ such that $\Upsilon(y,n)$ enumerates a dense subset of the (weak*-compact) set of completely positive maps from A into $M_n(\mathbb{C})$. Note that order zero maps form a closed subset of the set of completely positive maps, and the proof from [265] provides a Borel enumeration of a countable dense set of completely positive order zero maps.
Again as in [265], we use the fact that a map ψ from $M_n(\mathbb{C})$to A is completely positive if and only if $\psi(x_{ij})\sum_{i,j} x_{i,j}\, a_{i,j}$ where $\left(a_{i,j}\right)$ is a positive element of $M_n(A)$ of norm ≤ 1. By Farah et al.[262] there is a Borel function $\Xi: \Gamma \to (\Gamma^{\,n\times n})^{\mathbb{N}}$ such that $\Xi(y)$ is an enumeration of a countable dense set of such $\left(a_{i,j}\right)$.
Inspection of (i)– (iii)in the definition of $\dim_{\mathrm{nuc}}\left(C^*(y)\right) \leq n$ reveals that the verification of these conditions is only required over countable subsets of the allowable $\phi, \psi,$ and a, subsets which are computed in a Borel manner from y using the maps $Y, \Xi,$ and y itself, respectively. It follows that the set of y for which $\dim_{\mathrm{nuc}}\left(C^*(y)\right) \leq n$ is Borel.
Corollary (4.1.34)[370]: (i) The relation $r_2 \subseteq \Gamma \times \mathbb{N} \times \mathbb{N}$ defined by

$$r_1(y^2-1, n+\epsilon, n) \Leftrightarrow proj(y^2-1)(n+\epsilon) \sim C^*(\gamma) proj(y^2-1)(n)$$

is Borel.
(ii) The reˆlation $Y_2 \subseteq \Gamma \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ defined by

$$r_2(y^2-1, n+\epsilon, n, k) \Leftrightarrow proj(y^2-1)(n+\epsilon) \oplus proj(y^2-1)(n) \sim M_2\left(C^*(y^2-1)\right) proj(y^2-1)(k) \oplus 0$$

is Borel.
Proof: To see (i), note that

$$r_1(y^2-1, n+\epsilon, n) \Leftrightarrow (\exists k) \| pk(y^2-1) pk(y^2-1)^* - \mathrm{proj}(y^2-1)(n+\epsilon) \| < \tfrac{1}{4} \wedge \| pk(y^2-1)^* pk(y^2-1) - \mathrm{proj}(y^2-1)(n) \| < \tfrac{1}{4}.$$

For (ii), note that for $n+\epsilon, n, k \in \mathbb{N}$ the maps $\Gamma \to M_2\left(B(H)\right)$

$$y^2-1 \mapsto \mathrm{proj}(y^2-1)(n+\epsilon) \oplus \mathrm{proj}(y^2-1)(n) \text{and y} \mapsto \mathrm{proj}(y^2-1)(k) \oplus 0$$

are Borel by farah et al. [262]. Thus,

$$r_2(n+\epsilon, n, k) \Leftrightarrow (\exists i) \left\| p_i\left(M_2(y^2-1)\right) p_i\left(M_2(y^2-1)\right)^* - \mathrm{proj}(y^2-1)(n+\epsilon) \oplus \mathrm{proj}(y^2-1)(n) \right\| < \frac{1}{4} \wedge \left\| p_i\left(M_2(y^2-1)\right) p_i\left(M_2(y^2-1)\right)^* - \mathrm{proj}(y^2-1)(k) \oplus 0 \right\| < \frac{1}{4}$$

gives a Borel definition of r_2.
Corollary (4.1.35)[370]: There is a Borel map $K_0: \Gamma \to G_{\mathrm{ord}}$ such that

$$K_0(y^2-1) \simeq \left(K_0\left(C^*(y^2-1)\right), K_0^+\left(C^*(y^2-1)\right)\right).$$

Proof: By Farah et al [262] the unitization $\tilde{C}^*(y^2-1)$ of $C^*(y^2-1)$ is obtained via a Borel function, and by the above proof so is $K_0\left(\tilde{C}^*(y^2-1)\right)$. Then $K_0 C^*(y^2-1)$ is isomorphic to the quotient of $K_0\left(\tilde{C}^*(y^2-1)\right)$by its subgroup generated by the image of the identity in $\tilde{C}^*(y^2-1)$.
Corollary (4.1.36)[370]: Let $(S, D) \in Cu_0$. Then $(a_n) \ll^{\nearrow} (a_n+\epsilon)$ in $D^{\nearrow}$ if and only if $[a_n] \leq [a_n+\epsilon]$ in $W(D) \cong S$.

Proof: Suppose that $(a_n) \ll^{\nearrow} (a_n+\epsilon)$. It follows that for each $n \in \mathbb{N}$ there is $m(n)$such that

$$a_n a_{m(n)} + \epsilon \ll a_{m(n)+1} + \epsilon.$$

The statement $[(a_n)] \leq [(a_n+\epsilon)]$ amounts to the existence of $(a_n+2\epsilon) \in D^{\nearrow}$ such hat $(a_n) \approx (a_n+2\epsilon)$ and $(a_n+2\epsilon) \ll^{\nearrow} (a_n+\epsilon)$. Here we can take $(a_n+2\epsilon) = (a_n)$, completing the forward implication.

Suppose, conversely, that $[(a_n)] \leq [(a_n+\epsilon)]$, so that there is some $(a_n+2\epsilon) \in D^{\nearrow}$ such that $(a_n) \cong (a_n+2\epsilon)$and $(a_n+2\epsilon) \leq (a_n+\epsilon)$.Since $(a_n)$and $(a_n+2\epsilon)$are cofinal in each other with respect to $\ll$, it is immediate that $(a_n) \leq^{\nearrow} (a_n+\epsilon)$.

Corollary (4.1.37)[370]: Let $(S,D) \in Cu_0$.Then $a \leq a+\epsilon$ in D if and only if $[\eta(a)] \leq [\eta(a+\epsilon)]$ in $W(D) \cong S$.

Proof: By Lemma (4.1.17), it is enough to prove that $\epsilon \geq 0$ iff $\eta(a) \leq^{\nearrow} \eta(a+\epsilon)$ in $D^{\nearrow}$.Suppose first that. $a \leq a+\epsilon$. The sequence $(\eta(a)_n)$,being cofinal with respect to $\ll$ in $\{a+2\epsilon \in D \mid a+2\epsilon \ll a\}$, has a supremum in S, namely, a itself. A similar statement holds for $a+\epsilon$. For any $n \in \mathbb{N}$, we have $\eta(a)_n \ll a$, and $\sup \eta(a+\epsilon)_m = a+\epsilon \geq a$. It follows that $\eta(a+\epsilon)_m \gg \eta(a)_n$ for all m sufficiently large, whence $[\eta(a)] \leq [\eta(a+\epsilon)]$, as desired.

Suppose, conversely, that $\eta(a) \leq^{\nearrow} \eta(a+\epsilon)$ in $D^{\nearrow}$. Since $\sup \eta(a)_n = a, \sup \eta(a+\epsilon)_m = a+\epsilon$, and fo11r each n there is m such that $\eta(a)_n \ll \eta(a+\epsilon)_m$, it is immediate that $\epsilon \geq 0$ in S.

Corollary (4.1.38)[370]: Let $(S_j, \mathrm{e}) \in Cu_u$, and let $D_j \subseteq S_j$ be a countable sup-subsemigroup of S_j containing e. It follows that, with respect to the common element $e, r(S_j,e) = r\,(D_j,e)$.

Proof: We suppress the e and write only $r(D_j)$and $r(S_j)$. It is clear that $r(D_j) \leq r(S_j)$. Given $\epsilon > 0$, we will prove $r(S_j) \leq rD_j + \epsilon$. Choose $n+\epsilon, n \in \mathbb{N}$ to satisfy

$$r(D_j) < \frac{n+\epsilon}{n} < r(D_j) + \epsilon$$

Let $x^2, y^2 \in S_j$ satisfy

$$(n+1) \times +(n+\epsilon)e \ \leq ny^2.$$

There are rapidly increasing sequences (x_k^2) and (y_k^2) in D_j having suprema x^2 and y^2, respectively. Since e is compact, so is $(n+\epsilon)e$, that is $(n+\epsilon)e \ll (n+\epsilon)e$. Since $x_k^2 \ll (n+1)$ for any k, we can use the fact that addition respects $\ll$ to conclude that

$$(n+1)x_k^2 + (n+\epsilon)e \ll (n+1)x^2 + (n+\epsilon)e \leq ny^2.$$

It follows that

$$(n+1)x_k^2 + (n+\epsilon)e \ll ny^2.$$

Since the operation of addition respects the operation of taking suprema, we have $\sup ny_l^2 = \ ny^2$, whence for some (and hence all larger) $l_k \in \mathbb{N}$, we have

$$(n+1)x_k^2 + (n+\epsilon)e \leq ny_{lk}^2.$$

Now since $\frac{n+\epsilon}{n} > r(D_j)$ we conclude that $X_k \leq Y_{\iota k}$. Taking suprema yields $X \leq Y$, proving that $r(S_j) \leq \frac{n+\epsilon}{n} > r(D_j) + \epsilon$, as desired.

Section (4.2): Automorphisms of Separable C^*-Algebras

If A is a separable C^*-algebra, the group $\text{Aut}(A)$of automorphisms of A is a Polish group with respect to the topology of pointwise norm convergence. An automorphism of A is called (multiplier) *inner* if it is induced by the action by conjugation of a unitary element of the multiplier algebra $M(A)$ of A. Inner automorphisms form a Borel normal subgroup $Inn(A)$ of the group of automorphisms of A. The relation of *unitary equivalence* of automorphisms of A is the coset equivalence relation on $\text{Aut}(A)$determined by $Inn(A)$. The main result presented here asserts that if A does not have continuous trace, then it is not possible to effectively classify the automorphisms of A up to unitary equivalence using countable structures as invariants; in particular this rules out classification by K-theoretic invariants. (The K-theoretic invariants of C^*-algebras were shown to be computable by a Borel function in [257]. Even though [257]. does not consider the K-theory of $*$_homomorphisms, it is not difficult to verify that the proof can be adapted to show that the computation of K-theory of $*$-homomorphisms is given by a *Borel functor*. The main ingredient of the proof is the fact that one can enumerate in a Borel fashion dense sequences of projecti ons and of unitary elements of the algebra and of all its amplifications [254].) We will show that the existence of an outer derivation on a C^*-algebra A is equivalent to a seemingly stronger statement, that we will refer to as Property AEP (see Definition (4.2.11)), implying in particular the existence of an outer derivable automorphismof A.

The notion of effective classification can be made precise by means of Borel reductions in the framework of descriptive set theory (see [264] and [259]). If E and E' are equivalence relations on standard Borel spaces X and X respectively, then a Borel reduction from E to F is a Borel function $f: X \to X'$such that for every $x, y \in X, xEy$if and only if $f(x)E'f(\text{y})$. The Borel function f witnesses an effective classification of the objects of X up to $E,$ with E'-equivalence classes of objects of X' as invariants. (In [254] and [257] the computation of most of the invariants in the theory of C^*-algebras is shown to be Borel.)

If E and F are, as before, equivalence relations on standard Borel spaces, then E is *Borel reducible*to F if there is a Borel reduction from E to F. This can be interpreted as a notion that allows one to compare the complexity of different equivalence relations. Some distinguished equivalence relations are used as benchmarks of complexity. Among these are the relation $=_Y$ of equality for elements of a Polish space Y, and the relation $\simeq_C$ of isomorphism within some class of countable structures C. If E is an equivalence relation on a standard Borel space $X,$ we say that:

(a) E is smooth (or the elements of X are concretely classifiable up to E) if E is Borel reducible to $=_Y$ for some Polish space Y;

(b) E is classifiable by countable structures (or the elements of X are classifiable by countable structures up to E) if E is Borel reducible to $\simeq_C$for some class C of countable structures.

A nontrivial example of smooth equivalence relation is the relation of unitary equivalence of irreducible representations of a Type I C^*-algebra — see [100]. Since all uncountable Polish spaces are Borel isomorphic to $\mathbb{R}$, the class of smooth equivalence relations includes only the equivalence relations that are effectively classifiable using real numbers as invariants. The class of equivalence relations that are classifiable by countable structures is much wider. In fact most classification results in mathematics involve some class of countable structures as invariants. Elliott's seminal classification of AF algebras by the ordered K_0group in [161] is of this sort, as well as the K-theoretical classification of purely infinite simple nuclear C^*-algebras in the UCT class obtained by Kirchberg and Phillips in [289] and [193]. In the last decade a number of natural equivalence relations arising in different areas of mathematics have been shown to be not classifiable by countable structures. For example the type of invariants that appears in the spectral theorem for normal operators transcend countable structures by a result of Kechris and Sofronidis [263]. The theory of turbulence, developed by Greg Hjorth in the second half of the 1990s, plays a key role in the proof of this and of many other analogous results.

Turbulence is a dynamic condition on a continuous action of a Polish group on a Polish space, implying that the associated orbit equivalence relation is not classifiable by countable structures. Many nonclassifiability results were established directly or indirectly using this criterion. Hjorth showed in [260] that the orbit equivalence relation of a turbulent Polish group action is Borel reducible to the relation of homeomorphism of compact spaces, which in turn is reducible to the relation of isomorphism of separable simple nuclear unital C^*-algebras by a result of Farah, Toms and Tornquist [254]. As a consequence these equivalence relations are not classifiable by countable structures.

We use Hjorth's theory of turbulence to prove the following theorem. (See Definition (4.2.3) for the notion of continuous trace C^*-algebra.)

Theorem(4.2.1)[278]: If A is a separable C^*-algebra that does not have continuous trace, then the automorphisms of A are not classifiable by countable structures up to unitary equivalence.

Theorem(4.2.1) strengthens , where the automorphisms of A are shown to be not concretely classifiable under the same assumptions on the C^*-algebra A. We will in fact show that the same conclusion holds even if one only considers the subgroup consisting of approximately inner automorphisms of $A,$ i.e. pointwise limits of inner automorphisms.

A particular implication of Theorem (4.2.1) is that it is not possible to classify the automorphisms of any separable C^*-algebra that does not have continuous trace up to unitary equivalence by Borel-computable K-theoretic invariants. This should be compared with the classification results of (sufficiently outer) automorphisms up to other natural equivalence relations, such as outer conjugacy; see [244]. Nakamura showed in [244] that aperiodic automorphisms of Kirchberg algebras are classified by their KK-classes up to outer conjugacy. Theorem 1.4 of [239] asserts that there is only one outer conjugacy class of uniformly aperiodic automorphisms of UHF algebras. These results were more recently generalized and expanded to classification of actions of $\mathbb{Z}^2$and $\mathbb{Z}^n$ up to outer conjugacy or cocycle conjugacy (see [294], [293], [287], and [295]).

Phillips and Raeburn obtained in [85] a cohomological classification of automorphisms of a C^*-algebra with continuous trace up to unitary equivalence. Such classification implies that if A has continuous trace and the spectrum of A is homotopy equivalent to a compact space, then the normal subgroup $\mathrm{Inn}(A)$of inner automorphisms is closed in $\mathrm{Aut}(A)$; see [60]. In particular (cf. [100]) this conclusion holds when A is unital and has continuous trace. It follows from a standard result in descriptive set theory —see [259] — that the automorphisms of A are concretely classifiable up to unitary equivalence if and only if $\mathrm{Inn}(A)$is a closed subgroup of $\mathrm{Aut}(A)$. Theorem 0.8 of [60] and Theorem (4.2.1) therefore imply the following dichotomy result:

Theorem(4.2.2)[278]: *If A is a separable unital C^*-algebra, then the following statements are equivalent:*

(i) the automorphisms of A are concretely classifiable up to unitary equivalence;

(ii) the automorphisms of A are classifiable by countable structures up to unitary equivalence;

(iii) A has continuous trace.

More generally the same result holds if A is a separable C^*-algebra with (not necessarily Hausdorff) compact spectrum. Without this hypothesis the implication $3 \Longrightarrow 1$of Theorem (4.2.2) does not hold, as pointed out in [60]. We do not know if the implication $3 \Longrightarrow 2$ holds for a not necessarily unital C^*-algebra A. This is commented on more extensively.

In particular Theorem (4.2.2) offers another characterization of unital C^*-algebras that have continuous trace, in addition to the classical Fell-Dixmier spectral condition (see [57], [282]) or the reformulation in terms of central sequences by Akemann and Pedersen; see [1].

The dichotomy in the Borel complexity of the relation of unitary equivalence of automorphisms of a unital C^*-algebra expressed by Theorem (4.2.2) should be compared with the analogous phenomenon concerning the relation of unitary equivalence of irreducible representations of a C^*-algebra A. It is a classical result of Glimm from [286] that such a relation is smooth if and only if A is Type I. It was proved in [267] and, independently,

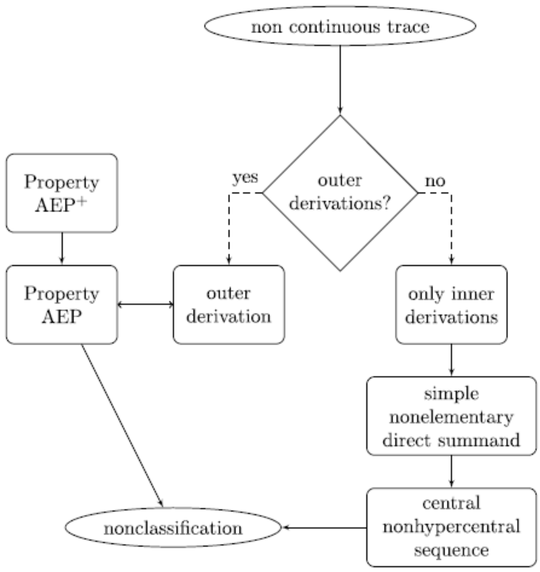


Fig. (1)[278]: Proof strategy. This diagram illustrates the strategy of the proof of Theorem (4.2.1).

In [283] that the irreducible representations of a C^*-algebra that is not Type I are in fact not classifiable by countable structures up to unitary equivalence.

The strategy of the proof of Theorem (4.2.1), summarized in Fig. (1), is the following: We first introduce in Definition (4.2.11) and (4.2.16) Properties AEP and AEP$^+$, named after Akemann, Elliott, and Pedersen since they can be found *in nuce* in their works [1] and [64]. (The main result of [64] is a characterization of C^*-algebras with only inner derivations as direct sum of simple C^*-algebras and C^*-algebras with no nontrivial central sequence [64]. Theorem 2.4 of [1] shows that a C^*-algebra is does not have any nontrivial central sequence if and only if it has continuous trace.) We then show in Proposition (4.2.17) that Property AEP$^+$ is stronger than Property AEP; moreover by Theorem (4.2.20) Property AEP is equivalent to the existence of an outer derivation, and by Lemma (4.2.10) it implies that the conclusion of Theorem (4.2.1) holds.

This concludes the proof under the assumption that the C^*-algebra A has an outer derivation. We then assume that A does not have continuous trace and has only inner derivations. Using the already mentioned characterization of C^*-algebras with only inner derivations from [64] and the characterization of continuous trace C^*-algebras in terms of central sequences given in [1], we infer that in this case A has a simple nonelementary direct summand. We then deduce in Proposition (4.2.25) that A contains a central sequence that is not strict-hypercentral. (A similar result was proved by Phillips in the unital case, cf. [298].) The proof is finished by proving that the existence of a central sequence that is not strict-hypercentral implies that the conclusion of Theorem (4.2.1) holds. This is done in Proposition (4.2.26).

Contains some background on C^*-algebras and introduces the notations used in the rest; infers from Hjorth's theory of turbulence a criterion of nonclassifiability by countable structures, to be applied in the proof of Theorem (4.2.1); establishes Theorem (4.2.1) in the case of C^*-algebras with outer derivations, while deals with the case of C^*-algebras with only inner derivations; present a dichotomy result for derivations analogous to Theorem (4.2.2).

We have tried to equally accessible to both set-theorists and operator-algebraists. A deep knowledge about operator algebras, see [297], [100], [83] and [296].

A C^*-algebra is a norm-closed self-adjoint subalgebra of the Banach $*_$algebra $B(H)$.of bounded linear operators on some Hilbert space H. The group $\mathrm{Aut}(A)$.of automorphisms of A is a Polish group with respect to the topology of pointwise convergence; see [60]. A C^*-algebra is called *unital* if it contains a multiplicative identity, usually denoted by 1. If A is unital and u is a unitary element of A (i.e. such that $uu^* = u^*u = 1$), then

$$\mathrm{Ad}(u)(x) = uxu^*$$

defines an automorphism $\mathrm{Ad}(u)$ of A. When A is not unital one can consider unitary elements of the multiplier algebra of A. The *multiplier algebra $M(A)$ofAis* the largest unital C^*-algebra containing A as an essential ideal; see [100]. It can be regarded as the noncommutative analog of the Stone-Cech compactification of a locally compact Hausdorff space. The strict topology on $M(A)$ is the locally convex vector space topology on $M(A)$ generated by the seminorm $x \mapsto \|ax\| + \|xa\| for\ a \in A$ [100]. A positive contraction b_0 of A is *strictly positive* if

$$ab_0^{\frac{1}{n}} \to a$$

for every $a \in A$[100]. If b_0 is any strictly positive contraction in A, then the strict topology on $M(A)$can be equivalently defined as the locally convex vector space topology on A generated by the single seminorm

$x \mapsto \|m_0x\| + \|xb_0\|$. The multiplier algebra of a separable C^*-algebra A is not norm separable (unless A is unital, in which case $M(A)$ coincides with A). Nonetheless the *strict topology*of $M(A)$ is Polish and induces a Polish group structure on the group $U(A)$of unitary elements of $M(A)$. If u is a unitary multiplier of A, i.e. an element of $U(A)$, then one can define as before the automorphism $\mathrm{Ad}(u)$of A. An automorphism of A is called *inner*if it is of the form $\mathrm{Ad}(u)$for some unitary multiplier u, and *outer* otherwise. Inner automorphisms of a separable C^*-algebra A form a Borel normal subgroup of $\mathrm{Aut}(u)$. Two automorphisms a and $\beta$of A are called *unitarily equivalent* $\alpha \circ \beta^{-1}$is inner or, equivalently,

$$\alpha(x) = \beta(uxu^*)$$

for some unitary multiplier u and every x GA. This defines a Borel equivalence relationon $\mathrm{Aut}(u)$.

A *representation*of a C^*-algebra A on a Hilbert space H is a $*$_homomorphism from $A$to the C^*-algebra $B(H)$of bounded linear operators on H; see [100]. Two representations $\pi, \pi'$of Aon Hilbert spaces $H, H'$are *unitarily equivalent*if there is a surjective linear isometry $U: H \to H'$such that

$U\pi(a) = \pi'(a)U$

for every $a \in A$. A representation n of A on a Hilbert space H is called *irreducible*if there is no nontrivial closed subspace of H which is $\pi(a)$-invariant for every $a \in A$. The *spectrum* $\hat{A}$ of a separable C^*-algebra A is the space of unitary equivalence classes of irreducible representations of A on a separable Hilbert space [83]. This is canonically endowed with the *hull-kernel topology*, which is the topology having as open basis the collection of sets of the form

$$\mathcal{O}_I = \{[\pi] \in \widehat{A} : I \subseteq \mathrm{Ker}(\pi)\}$$

for some closed ideal I of A. In general this topology has very poor separation properties, and can even fail to be T_0. Aclosed ideal of A is *primitive* if it is the kernel of an irreducible representation of A. A C^*-algebra A is called *primitive* if $\{0\}$ is a primitive ideal in A, i.e. A has a faithful irreducible representation. The *primitive spectrum* $\check{A}$of A is the space of primitive ideals of A endowed with the quotient topology from the canonical surjection

$$\hat{A} \to \check{A}$$
$$[\pi] \mapsto \mathrm{Ker}(\pi).$$

An element x of a C^*-algebra A is *abelian*if the closure of x^*Ax in A is a commutative subalgebra.

Definition(4.2.3)[278]: A separable C^*-algebra A has *continuous trace* if it is generated by abelian elements, and the spectrum $\hat{A}$ endowed with the hull-kernel topology is a Hausdorff space.

Equivalent reformulations of the notion of continuous trace C^*-algebras can be found in [100]. The class of C^*-algebras that do not have continuous trace is fairly large, and in particular includes all C^*-algebras that are not Type I. (A C^*-algebra A is Type I if every nonzero quotient of A contains a nonzero abelian element. Several equivalent characterizations of Type I C^*-algebras are listed in [100].) More information about C^*-algebras with continuous trace can be found in the monograph [299].

We assume all C^*-algebras to be norm separable, apart from multiplier algebras and enveloping von Neumann algebras. If A is a C^*-algebra, then the *universal representation* $\pi_u$of A is the direct sum of all cyclic representations of A associated with states of A [83]. The *enveloping von Neumann algebra* of A is the closure of $\pi_u[A]$in the strong operator topology. It is a well known theorem (see [83]) that the enveloping von Neumann algebra of A is isometrically isomorphic — as a Banach space —to the second dual of A. We will therefore denote in the following by A^{**} the enveloping von Neumann algebra of A. The σ-*weak topology* on A^{**} coincides with the weak* topology of A^{**} regarded as the dual Banach space of A^*. The algebra A can be identified with a σ-weakly dense subalgebra of A^{**}.Moreover by [83] we can identify the multiplier algebra $M(A)$ of$A$with the idealizer of A inside A^{**};i.e. the algebra of elements $x$such that $xa \in A$ and $ax \in A$ for every $a \in A$.

Analogously, the *unitization* $\tilde{A}$ of A [100] is identified with the subalgebra of $M(A)$ generated by A and 1.
If $x$is a normal element of A, i.e. commuting with its adjoint, and f is a complex-valued continuous function defined on the spectrum of x, then $f(\mathrm{x})$ denotes the element of A obtained from x and f using functional calculus (II.2 of [100]). If $x, y$are element of a C^*-algebra, then $[x, y]$denotes their *commutator*$xy - yx$; moreover if $S$is a subset of a C^*-algebra A, then $S' \cap A$denotes the *relative commutant*of S in A; see [100]. The set $\mathbb{N}$ of natural numbers is supposed not to contain 0. Boldface letters t and $s$indicate sequences of real numbers whose n-th terms are $t_n$and s_n respectively. Analogously x stands for the sequence $(x_n)_{n\in\mathbb{N}}$of elements of a C^*-algebra A.

Recall that a subset A of a Polish space X has the Baire property [264] if its symmetric difference with some open set is meager. A function between Polish spaces is Baire measurable [264] if the inverse image of any open set has the Baire property. Observe that, in particular, any Borel function is Baire measurable. Suppose that E and R are equivalence relations on Polish spaces X and Y, respectively. We say that E is *generically R-ergodic* if, for every Baire measurable function $f: X \to Y$ such that $f(x)Rf(\mathrm{y})$ whenever $xE\mathrm{y}$, there is a comeager subset C of X such that $f(x)Rf(\mathrm{y})$ for every $x, \mathrm{y} \in C$[259]. Observe that if E is generically R-ergodic and no equivalence class of E is comeager then, in particular, E is not Borel reducible to R.

The study of Borel complexity of equivalence relations is Hjorth's theory of turbulence. See [260]. Turbulence is a dynamical property of a continuous group action of a Polish group G on a Polish space X; see [260]. The main result about turbulent actions is the following result of Hjorth (Theorem 3.21 in [260]):

The orbit equivalence relation E_G^xassociated with a turbulent action $G \curvearrowright X$ of a Polish group G on a Polish space X is generically $\simeq_{\mathrm{C}}$-ergodic for every class C of countable structures, where $\simeq_{\mathrm{C}}$ denotes the relation of isomorphism for elements of C. Since (by definition of turbulence) E_G^x hasmeager equivalence classes, it is in particular not classifiable by countable structures.

This result is valuable because it allows one to obtain several nonclassification results. In order to apply such result it will be useful to first state and prove the following to easy lemmas:

Lemma(4.2.4)[278]: *Suppose that E, F, and R are equivalence relations on Polish spaces X, Y, and Z, respectively, and that F is generically R-ergodic. If there is a comeager subset C of Y and a Baire measurable function $f: \widetilde{C} \to K$ such that:*

(a) *$f(x)E(y)$ for any $x, y \in \widetilde{C}$ such that xFy;*

(b) *$f(C)$ is comeager in X for every comeager subset C of $\widetilde{C}$;*

then the relation E is generically R-ergodic as well.

Proof: Suppose that $g: X \to Z$is a Baire measurable function such that $g(x)\mathrm{R}g(x')$ for any $x, x' \in X$ such that xEx'. The composition $g\ of$ is a Baire measurable function from $\widetilde{C}$ to Z such that $(g\ of)(y)R(g\ of)(y')$for any $y, y' \in C$ such that yEy'.. Since $\widetilde{C}$ is comeager in Y, and F is generically R-ergodic, there is a comeager subset C of $\widetilde{C}$ such that $(g\ of)(y)R(g\ of)(y')$for every $y, y' \in C$. Therefore, $f[C]$is a comeager subset of X such that $g(x)\mathrm{R}g(x')$for every $x, x' \in f[C]$.

Observe that if f is continuous, open, and onto, then it will automatically satisfy the second condition of Lemma (4.2.4).

Lemma(4.2.5)[278]: *Suppose that E and F are equivalence relations on Polish spaces X and Y, respectively, and F is generically $\simeq C$-ergodic for every class C of countable structures. If there is a Baire measurable function $f: Y \to X$such that*

(a) $f(x)Ef(y)$ *whenever xFy, and*

(b) *no preimage of an E-class is comeager,*

then the relation E is not classifiable by countable structures.

Proof: Suppose by contradiction that there is a class C of countable structures and a Borel reduction $g: X \to C$ of E to $\simeq_C$ The composition $g \, of: Y \to C$ is a Baire measurable function from $Y$to C such that $(gof)(y) \simeq C(g\ o\ f)(y')$for any $y, y' \in Y$ such that yEy'. Since F is generically $\simeq_C$-ergodic, there is a comeager subset C of Y such that $(gof)(y) \simeq_C (g\ o\ f)(y')$ for every $y, y' \in C$. Therefore, being g a reduction of E to $\simeq_C$, $f(y)Ef(y')$for every $y, y' \in C$. This contradicts our assumptions .

Consider $\mathbb{R}^{\mathbb{N}}$ as a Polish space with the product topology and ℓ^1as a Polish group with its Banach space topology. The fact that the action of ℓ^1on $\mathbb{R}^{\mathbb{N}}$ by translation is turbulent is a particular case of [260]. It then follows by Hjorth's turbulence theorem that the associated orbit equivalence relation $E_{\mathbb{R}^{\mathbb{N}}}^{\ell^1}$ is generically $\simeq_C$-ergodic for every class C of countable structures. It is not difficult to see that the function $f: (\mathbb{R} \setminus \{0\})^{\mathbb{N}} \to (0,1)^{\mathbb{N}}$,defined by

$$f(t) = \left(\frac{|t_n|}{|t_n| + 1}\right)_{n\in\mathbb{N}}$$

satisfies both the first (being continuous, open, and onto) and the second condition of Lemma (4.2.4), where:

(a) F is the relation $E_{\mathbb{R}^{\mathbb{N}}}^{\ell^1}$of equivalence modulo $\ell^1$of sequences of real numbers;

(b) $E$is the relation $E_{\mathbb{R}^{\mathbb{N}}}^{\ell^1}$of equivalence modulo $\ell^1$of sequences of real numbers between 0 and 1.

It follows that the latter relation is generically $\simeq C$-ergodic for every class C of countable structures. Considering the particular case of Lemma (4.2.5) when F is the relation $E_{(0,1)^{\mathbb{N}}}^{\ell^1}$one obtains the following nonclassifiability criterion:

Criterion(4.2.6)[278]: If E is an equivalence relation on a Polish space X and there is a Baire measurable function $f: (0,1)^{\mathbb{N}} \to X$ such that:

(a) $f(x)Ef(y)$for any $x, y \in (0,1)^{\mathbb{N}}$,such that $x - y \in \ell^1$;

(b) any comeager subset of $(0,1)^{\mathbb{N}}$ contains elements x, y such that then the relation E is not classifiable by countable structures.

In order to apply Criterion (4.2.6). we will need the following fact about nonmeager subsets of $(0,1)^{\mathbb{N}}$:

Lemma(4.2.7)[278]: *If X is a nonmeager subset of $(0,1)^{\mathbb{N}}$,then there is an uncountable $Y \subset X$ such that, for every pair of distinct points t, s of Y, $\| s - t\|_\infty \geq \frac{1}{4}$, where*

$$\|t - s\|_\infty = \sup_{n\in\mathbb{N}} |t_n - s_n|.$$

Proof: Define for every $s \in (0,1)$,

$$K_s = \left\{ t \in (0,1)^{\mathbb{N}} \middle| \|t - s\|_\infty \leq \frac{1}{4} \right\}.$$

Observe that $K_s$is a closed nowhere dense subset of $(0,1)^{\mathbb{N}}$. Consider the class $\mathcal{A}$ of subsets Y of X with the property that, for every s, t in Y distinct, $\|s - t\| \geq \frac{1}{4}$. If $\mathcal{A}$ is partially ordered by inclusion, then it has some maximal element Y by Zorn's lemma. By maximality,

$$X \subset \bigcup_{t \in Y} \left\{ s \in (0,1)^{\mathbb{N}} \middle| \|t - s\|_\infty \leq \frac{1}{4} \right\}.$$

Since the set X is nonmeager, Y is uncountable.

The aim is to show that if a C^*-algebra A has an outer derivation, then the relation of unitary equivalence of approximately inner automorphisms of A is not classifiable by countable structures. In proving this fact we will also show that any such C^*-algebra satisfies a seemingly stronger property, that we will refer to as Property AEP (see Definition (4.2.13)).

A *derivation* of a C^*-algebra A is a linear function

$$\delta: A \to A$$

satisfying the *derivation identity*:

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for $x, y \in A$. The derivation identity implies that δ is a bounded linear operator on A; see [83]. The set $\Delta(A)$ of derivations of A is a closed subspace of the Banach space $B(A)$ of bounded linear operators on A. A derivation is called a $*_$derivation if it is a positive linear operator, i.e. it sends positive elements to positive elements. Any element $a$of the multiplier algebra of Adefines a derivation $\mathrm{ad}(ia)$of A,by

$$a\mathrm{d}(ia)(x) = [ia, x].$$

This is a $*_$derivation if and only if a is self-adjoint. A derivation of this form is called *inner*, and *outer* otherwise. More generally, if a is an element of the enveloping von Neumann algebra of A that *derives* A, .ie. $ax - xa \in A$ for any $x \in A$,then one can define the (not necessarily inner) derivation $\mathrm{ad}(ia)$ of A. Since any derivation is linear combination of $*_$derivations (see [83]), the existence of an outer derivation is equivalent to the existence of an outer $*_$derivation. The set $\Delta_0(A)$ of inner derivations of A is a Borel (not necessarily closed) subspace of $\Delta(A)$. The norm on $\Delta_0(A)$ defined by

$$\|a\mathrm{d}(ia)\|_{\Delta_0(A)} = \inf\{\|a - z\| \mid z \in Z(A)\},$$

where $Z(A)$ denotes the *center* of A, makes $\Delta_0(A)$ a separable Banach space isomet-rically isomorphic to the quotient of A by $Z(A)$ The inclusion of $\Delta_0(A)$in $\Delta(A)$is continuous, and the closure $\overline{\Delta_0(A)}$ of $\Delta_0(A)$ in $\Delta(A)$.is a closed separable subspace of $\Delta_0(A)$ If δ is a $*\_$ derivation then the exponential $exp(\delta)$ of δ, regarded as an element of the Banach algebra $B(A)$of bounded linear operators of A, is an automorphism of A. Automorphisms of this form are called derivable. If $\delta = a\mathrm{d}(ia)$is inner then

$$\exp(\delta) = Ad\big(exp(ia)\big)$$

is inner as well. Lemma (4.2.7) provides a partial converse to this statement. (The converse is in fact false in general; see [42].) For more information on derivations and derivable automorphisms, see[83].

Lemma(4.2.8)[278]: Suppose that A is a primitive C^*-algebra. If δ is a $*_$derivation of A with operator norm strictly smaller than 2π such that $exp(\delta)$ is inner, then δ is inner.
The lemma is proved in [42] under the additional assumption that A is unital. It is not difficult to check that the same proof works without change in the nonunital case.
Definition(4.2.9)[278]: Suppose that A is a C^*-algebra, $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in the unit ball of A, and $x = (x_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal positive contractions of A such that for every $n \in \mathbb{N}$ and $i \leq n$,

$$\|[x_n, a_i]\| \leq 2^{-n}. \tag{1}$$

Since the x_n $'s$ are pairwise orthogonal, if t is a sequence of real numbers of absolute value at most 1, then the series

$$\sum_{n\in\mathbb{N}} t_n x_n$$

converges in the strong operator topology to a self-adjoint element of A^{**}. Moreover, the sequence of inner automorphisms

$$\left(Ad\left(exp\left(i\sum_{k\leq\mathbb{N}}(t_k x_n)\right)\right)\right)_{n\in\mathbb{N}}$$

of A converges—in view of(1)—to the approximately inner automorphism

$$\alpha_t := Ad\left(exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)t_n x_n\right)\right)$$

The equivalence relation E_x on $(0,1)^{\mathbb{N}}$ is defined by

sE_xt iff α_t and $\alpha_s$are unitarily equivalent.

Observe that this equivalence relation is finer than the relation of ℓ^1-equivalence introduced. In fact if $s,t \in (0,1)^{\mathbb{N}}$ and $s,t \in \ell^1$, then the series

$$\sum_{n\in\mathbb{N}}(t_n - s_n)x_n$$

converges in A. It is then easily verified that

$$u := exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right)$$

is a unitary multiplier of A such that

$$\mathrm{Ad}(u)o\alpha_s = \alpha_t.$$

Therefore, if the equivalence classes of E_x are meager, the continuous function

$$(0,1)^{\mathbb{N}} \to \mathrm{Aut}(A)$$
$$t \longmapsto \alpha_t$$

satisfies the hypothesis of Criterion (4.2.6). This concludes the proof of the following lemma:
Lemma(4.2.10)[278]: *Suppose that A is a C^*-algebra. If for some sequence x of pairwise orthogonal positive contractions of A satisfying the commutation condition* (1) *the*

equivalence relation E_x has meager equivalence classes, then the approximately inner automorphisms of A are not classifiable by countable structures.

Lemma (4.2.10) motivates the following definition.

Definition(4.2.11)[278]: A C^*-algebra A has Property AEP if for every dense sequence $(a_n)_{n\in\mathbb{N}}$ in the unit ball of A there is a sequence $x = (x_n)_{n\in\mathbb{N}}$ of pairwise orthogonal positive contractions of A such that:

(i) $\|[x_n - a_i]\| < 2^{-\mathrm{n}}$ for $i \in \{1,2,\dots,n\}$;

(ii) the relation E_x as in Definition (4.2.9) has meager conjugacy classes.

It is clear that if a C^*-algebra A has Property AEP, then A has an outer $*_$derivation. In fact, if $s, t \in (0,1)^{\mathbb{N}}$ are such that $sE_x\, t$, then the self-adjoint element

$$a = \sum_{n\in\mathbb{N}} (t_n - s_n)x_n$$

of A^{**} derives A. The automorphism $\mathrm{Ad}(\exp(ia))$ is outer, and hence such is the $*_$derivation $\mathrm{ad}(ia)$. The rest is devoted to prove that, conversely, if A has an outer derivation, then A has Property AEP.

The following lemma shows that primitive nonsimple C^*-algebras have Property AEP. The main ingredients of the proof are borrowed from [64] and [1].

Lemma(4.2.12)[278]: *If A is a primitive nonsimple infinite-dimensional C^*-algebra, then it has Property* AEP.

Proof: Fix a faithful irreducible representation $\pi : A \to B(H)$. By [83] π extends to a σ-weakly continuous representation $\pi^{**}: A^{**} \to B(H)$. Fix a dense sequence $(a_n)_{n\in\mathbb{N}}$ in the unit ball of A and a strictly positive contraction b_0 of A. (Recall that a positive contraction b_0 of A is strictly positive if

$$ab_0^{\frac{1}{n}} \to a$$

for every $a \in A$ [100].) As in the proof of [1], one can define a sequence $x = (x_n)_{n\in\mathbb{N}}$ of pairwise orthogonal projections such that for some $\varepsilon > 0$ and every $k, n \in \mathbb{N}$ such that $k \leq n$,

(a) $\|x_n b_0\| > \varepsilon$;

(b) $\|x_n\, , a_k\| < 2^{-n}$.

Now suppose by contradiction that the equivalence relation E_x has a nonmeager equivalence class X. Thus for every $t\,, s \in X$ the automorphism

$$\alpha_{t\,,s} = \mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right)\right)$$

is inner. Fix $t\,, s \in X$. Observe that $\alpha_{t\,,s}$ is the exponential of the $*_$derivation

$$\delta_{t\,,s} = \mathrm{ad}\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right).$$

By Lemma (4.2.8) the $*_$derivation $\delta_{t\,,s}$ is inner. Thus, there is an element $z_{t\,,s}$ of the center of the enveloping von Neumann algebra of A such that

$$\sum_{n\in\mathbb{N}}(t_n - s_n)x_n + z_{t,s} \in M(A).$$

Recall that π has been extended to a σ-weakly continuous representation $\pi^{**}: A^{**} \to B(H)$ by [83]. The image of a central element of A^{**} under π^{**} belongs to the relative commutant of $\pi[A]$ in $B(H)$, which consists only of scalar multiples of the identity by [100]. Thus,

$$\pi\left(\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right) \in \pi^{**}[M(A)].$$

Hence

$$\pi\left(b_0\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right) \in \pi[A].$$

By Lemma (4.2.7) one can find an uncountable subset Y of X such that any pair of distinct elements of Y has uniform distance at least $\frac{1}{4}$. Fix $s \in Y$. For all $t, t' \in Y,$ there is $m \in \mathbb{N}$ such that

$$|t_m - t'_m| \geq \frac{1}{4}.$$

Henceforth,

$$\begin{aligned}\left\|\pi\left(b_0\left(\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right)\right) - \pi\left(b_0\left(\sum_{n\in\mathbb{N}}(t'_n - s_n)x_n\right)\right)\right\| &= \left\|\pi\left(b_0\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right)\right\| \\ = \left\|b_0\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right\| &\geq \left\|b_0\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n x_m a_0\right\| \\ \geq |t_m - t'_m|\|(x_m b_0)^*(x_m b_0)\| &\geq \frac{\varepsilon^2}{4}.\end{aligned}$$

Since Y is uncountable, this contradicts the separability of $\pi[A]$.

In order to prove Property AEP for all C^*-algebra with outer $*$-derivations we need the fact that Property AEP is *liftable*. This means that if a $*$-homomorphic image of a C^*-algebra A has Property AEP, then A has Property AEP. (See Chapter 8 of [292].)

Lemma(4.2.13)[278]: *If $\pi: A \to B$ is a surjective $*$-homomorphism and B has Property AEP, then A has Property AEP.*

Proof: Suppose that $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in AThus, $(\pi(a_n))_{n\in\mathbb{N}}$is a dense sequence in B. Pick a sequence $(y_n)_{n\in\mathbb{N}}$, in B obtained from $(\pi(a_n))_{n\in\mathbb{N}}$ as in the definition of Property AEP. By [292], there is a sequence $(z_n)_{n\in\mathbb{N}}$ of pairwise orthogonal positive contractions of A such that $\pi(z_n) = y_n$ for every $n \in \mathbb{N}$. Fix an increasing quasicentral approximate unit of $\mathrm{Ker}(\pi)$ (cf. [100]), i.e. a sequence $(u_k)_{n\in\mathbb{N}}$ of elements of $\mathrm{Ker}(\pi)$ such that:

(a) $lim_{k\to+\infty}\|u_k x - x\| = lim_{k\to+\infty}\|x u_k - x\| = 0$ for every $x \in \ker(\pi)$;

(b) $lim_{k\to+\infty}\|[u_k, a]\| = 0$ for every $a \in A$.

For every $n, i \in \mathbb{N}$ such that $i \leq n,$by [100],

$$\lim_{k\to+\infty}\left\|z_n^{\frac{1}{2}}(1-u_k)z_n^{\frac{1}{2}}a_i - a_i z_n^{\frac{1}{2}}(1-u_k)z_n^{\frac{1}{2}}\right\| = \lim_{k\to+\infty}\|(1-u_k)(z_n a_i - a_i z_n)\| = \|y_n\pi(a_i - \pi(a_i)y_n)\| < 2^{-n}.$$

Thus, there is $k_n \in \mathbb{N}$ such that, if

$$x_n = z_n^{\frac{1}{2}}(1-u_{k_n})z_n^{\frac{1}{2}},$$

then

$$\|x_n a_i - a_i x_n\| < 2^{-n}$$

for every $i \leq n$. Observe that $(x_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal positive contractions of A. Moreover, if $E \subset (0,1)^{\mathbb{N}}$ is nonmeager, consider $s, t \in E$, such that the automorphism

$$\mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)\,y_n\right)\right).$$

of B is outer. We claim that the automorphism

$$\mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)\,x_n\right)\right).$$

of A is outer. Suppose that this is not the case. Thus, there is z in the center of the enveloping von Neumann algebra of A such that

$$\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)\,x_n\right) + z \in U(A).$$

Denoting by $\pi^{**}: A^{**} \to B^{**}$ the normal extension of π— see [100] — one has that

$$\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)\,y_n\right) + \pi^{**}(z) = \pi^{**}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)\,x_n\right) + z\right) \in U(B)$$

by Theorem 4.2 of [3]. Since $\pi^{**}(z)$ belongs to the center of the enveloping von Neumann algebra of B,

$$\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)\,y_n\right) + \pi^{**}(z)$$

is a unitary multiplier of B that implements

$$\mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)\,y_n\right)\right).$$

Hence, the latter automorphism of B is inner, contradicting the assumption.

Liftability of Property AEP allows one to easily bootstrap Property AEP from primitive nonsimple C^*-algebras to C^*-algebra whose primitive spectrum is not T_1.

Lemma(4.2.14)[278]: *If A is a C^*-algebra whose primitive spectrum $\check{A}$ is not T_1, then A has Property* AEP.

Proof: Since $\check{A}$ is not T_1, by [83] there is an irreducible representation π of A whose kernel is not a maximal ideal. This implies that the image of A under π is a nonsimple primitive C^*-algebra. By Lemma (4.2.13) the latter C^*-algebra has Property AEP. Therefore, being Property AEP liftable by Lemma (4.2.14), A has Property AEP.
In order to show that a C^*-algebra A has Property AEP, it is sometimes easier to show that it has a stronger property that we will refer to as Property AEP$^+$. Property AEP$^+$ appears, without being explicitly defined, and the main theorem of [64], as well as in the proofs of Lemma 3.5 and Lemma 3.6 of [1].
Recall that a bounded sequence $(x_n)_{n\in\mathbb{N}}$, of elements of A is called central if for every $a \in A$,

$$\min_{n\to+\infty} \|[x_n, a]\|$$

The beginning of contains a discussion about the notion of central sequence, the related notion of hypercentral sequence, and their basic properties.

Definition(4.2.15)[278]: A C^*-algebra A has Property AEP$^+$if there is a sequence $(\pi_n)_{n\in\mathbb{N}}$ of irreducible representations of A such that, for some positive contraction b_0 of A and a central sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise orthogonal positive contractions of A:

(a) the sequence

$$\left(\pi_n\big((x_n-\lambda)b_0\big)\right)_{n\in\mathbb{N}}$$

does not converge to 0 for any $\lambda \in \mathbb{C}$;
(b) $x_m \in \mathrm{Ker}(\pi_n)$ for every pair of distinct natural numbers n, m.
To show that Property AEP$^+$ implies Property AEP we will need the following lemma:
Lemma(4.2.16)[278]: Fix a strictly positive real number η. For every $\varepsilon > 0$ there is $\delta > 0$ such that for every C^*-algebra A and every pair of positive contractions x, b of A such that $\|b\| \geq \eta$, if

$$\|(exp(i\,x) - \mu)b\| \leq \delta$$

for some $\mu \in \mathbb{C}$ then

$$\|(x-\lambda)b\| \leq \varepsilon$$

for some $\lambda \in \mathbb{C}$.
Proof: Fix $\varepsilon > 0$. Let L be the principal branch of the logarithm. Since L is an analytic function on the open disc of radius 1 centered in 1, there is a polynomial

$$p(Z) = \rho_0, \rho_1 Z + \cdots + \rho_n Z^n$$

Such that

$$|p(z) - L(z)| \leq \frac{\varepsilon}{2}$$

for every $z \in \mathbb{C}$ such that $|z-1| \leq \exp(i)$. In particular for every $t \in [0,1]$

$$\big|p\big(\exp(it)\big) - \mathrm{t}\big| = \big|p\big(\exp(it)\big) - \mathrm{L}\big(\exp(it)\big)\big| \leq \frac{\varepsilon}{2}.$$

If $\mu \in \mathbb{C}$ is such that $|\mu| \leq \frac{2}{\eta}$,define $\rho_\mu(z)$ to be the polynomial in Z obtained from $p(Z)$ by replacing the indeterminate Z by $Z + \mu$. Observe that the j-th coefficient of $\rho_\mu(z)$ is

$$\rho_j^\mu = \sum_{i=j}^{n} |\rho_i| \binom{i}{j} \mu^{j-i}$$

for $0 \le j \le n$. Finally define

$$C = \sum_{1\le j\le i\le n} |\rho_i| \binom{i}{j} \left(\frac{3}{\eta}\right)^{j-i} \left(\frac{2}{\eta}\right)^{j-1}$$

and

$$\delta = min\left\{\frac{\varepsilon}{2C}, 1\right\}.$$

Suppose that A is a C^*-algebra and $x, b \in A$ are positive contractions such that $\|b\| \ge \eta$ and, for some $\mu \in \mathbb{C}$,

$$\|(\exp(ix) - \mu)b\| \le \delta.$$

Thus,

$$|\mu| \le \frac{2}{\eta}.$$

Moreover

$$\begin{aligned}\left\|\left(x - \rho_0^{\mu}\right)b\right\| &= \left\|\left(p\left(\exp(ix)\right) - \rho_0^{\mu}\right)b\right\| + \frac{\varepsilon}{2} = \left\|\left(\sum_{j=1}^{n} \rho_j^{\mu} \left(\exp(ix) - \mu\right)^j\right)b\right\| + \frac{\varepsilon}{2} \\ &\le \sum_{j=1}^{n} \left|\rho_j^{\mu}\right| \|\exp(ix) - \mu\|^{j-1}\, \delta + \frac{\varepsilon}{2} \le \sum_{j=1}^{n}\sum_{i=j}^{n} |\rho_i| \binom{i}{j} \left(\frac{2}{\eta}\right)^{j-i} \left(\frac{3}{\eta}\right)^{j-1} \delta + \frac{\varepsilon}{2} \\ &\le C\delta + \frac{\varepsilon}{2} \le \varepsilon.\end{aligned}$$

This concludes the proof.

We can now show that Property AEP^+ implies Property AEP.

Proposition(4.2.17)[278]: *If a C^*-algebra A has Property* AEP^+, *then it has Property* AEP.

Proof: Suppose that $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of irreducible representations of A, b_0 is a positive contraction of A of norm 1, and $(x_n)_{n\in\mathbb{N}}$ is a sequence of orthogonal positive elements of A as in the definition of Property AEP^+. Fix a dense sequence $(a_n)_{n\in\mathbb{N}}$ in the unit ball of A. After passing to a subsequence of the sequence $(x_n)_{n\in\mathbb{N}}$, we can assume that for some $\varepsilon > 0$, for every $\mu \in \mathbb{C}$ and every $n \in \mathbb{N}$,

$$\left\|\pi_n\left((x_n - \mu)b_0\right)\right\| \ge \varepsilon$$

and

$$\|[x_n, a_i]\| < 2^{-n}$$

for $i > n$. Thus, for every $\mu \in \mathbb{C}, n \in \mathbb{N}$ and $t \in \left(\frac{1}{4}, 1\right)$,

$$\left\|\pi_n\left((tx_n - \mu)b_0\right)\right\| \ge \frac{\varepsilon}{4}. \qquad (2)$$

Observe that, in particular,

$$\|\pi_n(b_0)\| \ge \varepsilon$$

for every $n \in \mathbb{N}$. Consider $\delta > 0$ obtained from $\frac{\varepsilon}{8}$ as in Lemma (4.2.17) (where we set $\eta = \varepsilon$). We claim that for every $t \in \left(\frac{1}{4}, 1\right), n \in \mathbb{N}$, and $\mu \in \mathbb{C}$,

$$\left\|\pi_n\left((\exp(itx_n) - \mu)b_0\right)\right\| < \delta.$$

In fact suppose by contradiction that there are $t \in \left(\frac{1}{4}, 1\right), n \in \mathbb{N}$, and $\mu \in \mathbb{C}$, such that

$$\left\|\left(\exp\left(it\pi_n(x_n)\right) - \mu\right)\pi_n(b_0)\right\| = \left\|\pi_n\left((\exp(itx_n) - \mu)b_0\right)\right\| < \delta.$$

Thus by our choice of δ there is $\mu \in \mathbb{C}$ such that

$$\left\|\pi_n\left((itx_n - \mu)b_0\right)\right\| = \|(it\pi_n(x_n) - \mu)\pi_n(b_0)\| \leq \frac{\varepsilon}{8}.$$

Such inequality contradicts (2).This concludes the proof of the assertion that for every $t \in \left(\frac{1}{4}, 1\right), n \in \mathbb{N}$, and $\mu \in \mathbb{C}$,

$$\left\|\pi_n\left((\exp(itx_n) - \mu)b_0\right)\right\| \geq \delta.$$

We now claim that the sequence $(x_n)_{n\in\mathbb{N}}$ witnesses the fact that A has Property AEP. Assume by contradiction that there is a nonmeager subset X of $(0,1)^{\mathbb{N}}$ such that for every $s, t \in X$, the automorphism

$$\mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right)\right)$$

of A is inner. If $s, t \in X$, then there is an element $z_{t,s}$ in the center of the enveloping von Neumann algebra of A such that

$$\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n + z_{t,s}\right)$$

multiplies A Hence,

$$y_{t,s} = \exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n + z_{t,s}\right)b_0$$

is an element of A. By Lemma (4.2.10), one can find an uncountable subset Y of X such that, for any $s, t \in Y$, there is $m \in \mathbb{N}$ such that

$$|t_m - s_m| \geq \frac{1}{4}.$$

Fix $s \in Y$, and observe that, for $t, t' \in Y$,

$$\pi_{no}\left(\exp\left(z_{t',s} - z_{t,s}\right)\right) = \mu 1$$

is a scalar multiple of the identity. Therefor

$$\begin{aligned}\left\|y_{t,s} - y_{t',s}\right\| &= \left\|\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right) - \exp\left(z_{t',s} - z_{t,s}\right)\right)a_0\right\| \\ &\geq \left\|\pi_{no}\left(\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right) - \exp\left(z_{t',s} - z_{t,s}\right)\right)a_0\right)\right\| \\ &= \left\|\pi_{no}\left(\left(\exp\left((t_{no} - t'_{no})x_n\right) - \mu\right)a_0\right)\right\| \geq \varepsilon.\end{aligned}$$

This contradicts the separability of A.

The proofs of Lemma (4.2.19) and Lemma (4.2.20) are contained, respectively, in the proofs of Lemmas 3.6 and 3.7 of [1] and in the proof of the implication $(i) \Rightarrow (ii)$ at page 139 of [64].

Recall that a point x of a topological space X is called *separated* if, given any point y of X distinct from x, there are disjoint open neighbourhoods of x and y.

Lemma(4.2.18)[278]: *Suppose that A is a C^*-algebra whose primitive spectrum $\check{A}$ is T_1. Consider a sequence $(\xi_n)_{n\in\mathbb{N}}$ of separated points in $\check{A}$. Define F to be the set of limit points of the sequence $(\xi_n)_{n\in\mathbb{N}}$* N *and I to be the closed self-adjoint ideal of A corresponding to F. If either the quotient A/I does not have continuous trace, or the multiplier algebra of A/I has nontrivial center, then A has Property* AEP^+.

Lemma(4.2.19)[278]: *If A is a C^*-algebra whose spectrum $\hat{A}$ is homeomorphic to the one-point compactification of a countable discrete space, then A has Property* AEP^+.

We can now show the following result that Property AEP as defined in Definition (4.2.11) is equivalent to having an outer $*$derivation.

Theorem(4.2.20)[278]: *If A is a C^*-algebra, the following statements are equivalent:*

(i) A has an outer derivation;

(ii) A has Property AEP.

Proof. We have already pointed out that Property AEP implies the existence of an outer $*$derivation. It remains only to show the converse. Suppose that A has an outer derivation. By [64], either there is a quotient B of A whose spectrum $\hat{B}$ is homeomorphic to the one point compactification of a countable discrete space, or the primitive spectrum $\check{A}$ of A is not Hausdorff. In the first case, A has Property AEP by virtue of Lemma (4.2.20) and Lemma (4.2.14). Suppose that, instead, the primitive spectrum $\check{A}$ of A is not Hausdorff. If $\check{A}$ is not even T_1, the conclusion follows from Lemma (4.2.15). Suppose now that $\check{A}$ is T_1. Since $\check{A}$ is not Hausdorff, there are two points ρ_0, ρ_1 of $\check{A}$ that do not admit any disjoint open neighbourhoods. By [281] the set of separated points is dense in $\check{A}$. Therefore can find a sequence $(\xi_n)_{n\in\mathbb{N}}$ of separated points of A whose set F of limit points contains both $\rho_0 and \rho_1$. Define I to be the closed self-adjoint ideal I of A corresponding to the closed subset F of $\check{A}$. Since F contains at least two points, A/I is nonsimple. Consider now two cases: If A/I has continuous trace then by [1] and [3], the multiplier algebra of A/I has nontrivial center. Therefore A has Property AEP^+ by Lemma (4.2.19). On the other hand if A/I does not have continuous trace, then again A has Property AEP^+ by Lemma (4.2.19). In either case, it follows that A has Property AEP^+ and, in particular, Property AEP.

We show that, if a C^*-algebra A with only inner derivations does not have continuous trace, then the relation of unitary equivalence of approximately inner automorphisms of A is not classifiable by countable structures. In proving this fact we will also show that any such C^*-algebra contains a central sequence that is not strict-hypercentral.

If A is a C^*-algebra, denote by A^∞ the quotient of the direct product $\prod_{n\in\mathbb{N}} A$ by the direct sum $\bigoplus_{n\in\mathbb{N}} A$; see [100]. Identifying as it is customary A with the algebra of elements of A^∞ admitting constant representative sequence, denote by A_∞ the relative commutant

$$(A' \cap A^\infty) = \{x \in A^\infty \ |\forall_y \in A, [x,y] = 0\}.$$

Finally define

$$\text{Ann}(A, A_\infty) = \{x \in A_\infty \ |\forall_y \in A, xy = 0\}$$

to be the annihilator ideal of A in A_∞. Observe that, if A is unital, then $\text{Ann}(A, A_\infty)$ is the trivial ideal.

A central sequence in a C^*-algebra A is a bounded sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A that asymptotically commute with any element of A. This means that for any $a \in A$,

$$\lim_{n\to+\infty} \|[x_n, a]\| = 0.$$
Equivalently the image of $(x_n)_{n\in\mathbb{N}}$ in the quotient of $\prod_{n\in\mathbb{N}} A$ by $\bigoplus_{n\in\mathbb{N}} A$ belongs to A_∞. From the last characterization it is clear that if $(x_n)_{n\in\mathbb{N}}$ is a central sequence of normal elements A with spectra contained in some subset D of $\mathbb{C}$ and $f: D \to \mathbb{C}$ is a continuous function such that $f(0) = 0$, then the sequence $\left(f(x_n)\right)_{n\in\mathbb{N}}$ is central. It is not difficult to verify that, if $(x_n)_{n\in\mathbb{N}}$ is a central sequence and $b \in M(A)$, then the sequence $([\,b, x_n])_{n\in\mathbb{N}}$ converges strictly to 0. (The strict topology on $M(A)$ has been defined.)
Let us call a central sequence $(x_n)_{n\in\mathbb{N}}$ *norm-hypercentral* if it asymptotically commutes in the norm topology with any other central sequence. This amounts to say that for any other central sequence $(y_n)_{n\in\mathbb{N}}$
$$\lim_{n\to+\infty} \|[x_n, y_n]\| = 0.$$
Equivalently the image of $(x_n)_{n\in\mathbb{N}}$ in the quotient of $\prod_{n\in\mathbb{N}} A$ by $\bigoplus_{n\in\mathbb{N}} A$ belongs to the center of A_∞. For our purposes it will be more convenient to look at central sequences that asymptotically commute *in the strict topology* with any other central sequence. This motivates the following definition:
Definition(4.2.21)[278]: Suppose that A is a C^*-algebra. A sequence $(x_n)_{n\in\mathbb{N}}$of elements of A is strict-hypercentral if it is central and, for any other central sequence $(y_n)_{n\in\mathbb{N}}$, the sequence
$$([x_n, y_n])_{n\in\mathbb{N}}$$
converges to 0 in the strict topology.

Observe that a central sequence $(x_n)_{n\in\mathbb{N}}$ is strict-hypercentral if and only if the image of the element of A_∞ having $(x_n)_{n\in\mathbb{N}}$ as representative sequence in the quotient $A_\infty / \mathrm{Ann}(A, A_\infty)$ belongs to the center of $A_\infty/\mathrm{Ann}(A, A_\infty)$. It is clear from this characterization that, if $(x_n)_{n\in\mathbb{N}}$ is a strict-hypercentral sequence of normal elements of A with spectra contained in some subset D of $\mathbb{C}$ and $f: D \to \mathbb{C}$ is a complex-valued continuous function such that $f(0) = 0$, then the sequence $\left(f(x_n)\right)_{n\in\mathbb{N}}$ is strict-hypercentral. When A is unital the ideal $\mathrm{Ann}(A, A_\infty)$ is trivial, and hence the notions of strict-hypercentral and norm-hypercentral sequence coincide. Therefore in the unital case a norm-hypercentral sequence will be simply called hypercentral.
The fact that a unital simple infinite-dimensional C^*-algebra contains a central sequence that is not hypercentral is a particular case of [298]. We will show here how one can generalize this fact to all simple nonelementary C^*-algebras. The proof deeply relies on ideas from [298].
Lemma(4.2.22)[278]: If $(x_n)_{n\in\mathbb{N}}$ is a strict-hypercentral sequence in A and α is an approximately inner automorphism of A, then $(\alpha(x_n) - x_n)_{n\in\mathbb{N}}$ converges strictly to 0.
Proof: The same proof of Kaplansky's density theorem [83] shows that the unit ball of A is strictly dense in the unit ball of $M(A)$; see [291]. (The strict topology on the multiplier algebra of A has been defined.) It follows that, if $\varepsilon > 0$ and a is an element of A, then there is a finite subset F of the unit ball of A, a positive real number δ, and a natural number n_0 such that, for every $n \geq n_0$ and every y in the unit ball $M(A)$such that $\|[y, z]\| \leq \varepsilon$ for every $z \in F$,,
$$\max\{\|a(x_n y - y x_n)\|, \|(x_n y - y x_n)a\|\} \leq \varepsilon.$$
Consider the open neighbourhood
$$U = \{\alpha \in \mathrm{Aut}(A) |\ \|\alpha(x) - x\| < \delta \text{ for every } x \in F\}$$

of id_A in $\mathrm{Aut}(A)$. Observe that if $\beta \in U$ is inner, then for every $n \geq n_n$

$$\|(\beta(x_n) - x_n)a\| \leq \varepsilon$$

and

$$\|a(\beta(x_n) - x_n)\| \leq \varepsilon.$$

Approximating with inner automorphisms, one can see that the same is true if $\beta \in U$ is just approximately inner. Since α is approximately inner, there is a unitary multiplier u of A and an approximately inner automorphism β of A in U such that

$$\alpha = \beta \ o \ \mathrm{Ad}(u).$$

Consider a natural number $n_1 \geq n_0$ such that, for $n \geq n_1$,

$$= \|\beta^{-1}(a)[x_n, u\ ]\| \leq \varepsilon$$

and

$$\|[x_n, u^*]\beta^{-1}(a)\| \leq \varepsilon.$$

It follows that, if $n \geq n_1$,

$$\begin{aligned}\|a(\alpha(x_n) - x_n)\| &\leq \left\|a\beta\left(\mathrm{Ad}(u)(x_n)\right) - x_n\right\| + \|\beta(z_n) - x_n\| \\ &\leq \|\beta^{-1}(a)(ux_nu^* - z_n)\| + \varepsilon \\ &= \|\beta^{-1}(a)[x_n, u\ ]\| + \varepsilon \\ &\leq 2\varepsilon\end{aligned}$$

and, analogously,

$$\|(\alpha(x_n) - x_n)a\| \leq 2\varepsilon.$$

Since ε was arbitrary, this concludes the proof of the fact that

$$(a(z_n) - x_n)_{n\in\mathbb{N}}$$

converges strictly to 0.

If α is an automorphism of a C^*-algebra A, then α^{**} denotes the unique extension of α to a σ-weakly continuous automorphism of the enveloping von Neumann algebra A^{**} of A (defined as in [100]).

Lemma(4.2.23)[278]: *Suppose that A is a C^*-algebra such that every central sequence in A is strict-hypercentral. If α is an approximately inner automorphism of A, then α^{**} fixes pointwise the center of A^{**}, i.e. $\alpha^{**}(z) = z$ for every central element of A^{**}.*

Proof: Observe that z derives A, since

$$za - az = 0 \in A.$$

for every $a \in A$. Thus, by Lemma 1.1 of [1], there is a bounded net (z_λ) in A converging strongly to z such that, for every $a \in A$,

$$\lim_{\lambda} \|[z_\lambda - z, a]\| = 0.$$

Recall that strong and σ-strong topology agree on bounded sets, and that the σ-strong topology is stronger than the σ-weak topology; see [100]. Thus the net (z_λ) converges a fortiori σ-weakly to z. Since the σ-weak topology on A^{**} is the weak* topology on A^{**} regarded as the dual space of A^*, the unit ball of A^{**} is σ-weakly compact by Alaoglu's theorem [297]. Moreover by Kaplansky's Density Theorem [83] the unit ball of A is σ-weakly dense in the unit ball of A^{**}. As a consequence the unit ball of A^{**} is σ-weakly metrizable, and the same holds for balls of arbitrary radius. Thus we can find a bounded *sequence* $(z_n)_{n\in\mathbb{N}}$ in A converging σ-weakly to z such that, for every $a \in A$,

$$\lim_{n\to+\infty} \|[z_n - z, a]\| = 0.$$

Since

$$[z_n - z, a] = [z_n, a]$$

for every $n \in \mathbb{N}$, $(z_n)_{n\in\mathbb{N}}$ is a central and hence strict-hypercentral sequence (every central sequence of A is strict-hypercentral by assumption). Since α^{**} is a σ-weakly continuous automorphism of A^{**} extending α, $\left(\alpha(z_n)\right)_{n\in\mathbb{N}}$ converges σ-weakly to $\alpha^{**}(z)$. It follows from Lemma (4.2.23) and from the facts that α is approximately inner and the sequence $(z_n)_{n\in\mathbb{N}}$ is strict-hypercentral that the bounded sequence $\left(z_n - \alpha(z_n)\right)_{n\in\mathbb{N}}$ converges strictly to 0. By [292] and since weak and σ-weak topology agree on bounded sets, the sequence $\left(z_n - \alpha(z_n)\right)_{n\in\mathbb{N}}$ converges σ-weakly to 0. Therefore $z = \alpha^{**}(z)$.

A C^*-algebra is called *elementary* if it is $*$_isomorphic to the algebra of compact operators on some Hilbert space; see [100]. By Corollary 1 of Theorem 1.4.2 in [279] any elementary C^*-algebra is simple. Moreover by Corollary 3 of Theorem 1.4.4 in [279] any automorphism of an elementary C^*-algebra is inner; in particular the group $\mathrm{Inn}(A)$of inner automorphisms of an elementary C^*-algebra A is closed inside the group $\mathrm{Aut}(A)$of all automorphisms. Conversely if the group of inner automorphisms of a simple C^*-algebra A is closed, then A is elementary by [100].

Recall that all C^*-algebras (apart from multiplier algebras and enveloping von Neumann algebras) are assumed to be norm separable. In particular separability of A is assumed in Proposition (4.2.24); however we do not know if the separability assumption is necessary there. (This is also asked in [284].)

Proposition(4.2.24)[278]: *If A is a simple C^*-algebra such that every central sequence in A is strict-hypercentral, then A is elementary.*

Proof: It is enough to show that $\mathrm{Inn}(A)$is closed in $\mathrm{Aut}(A)$or, equivalently, that no outer automorphism is approximately inner. Fix an outer automorphism α of A. Since A is simple, by [290], there is an irreducible representation π such that π and $\pi\ o\ \alpha$ are not unitarily equivalent. If z is the central cover of π in A^{**} (defined as in [83]), then $\alpha^{**}(z)$ is the central cover of $\pi\ o\ \alpha$ moreover, being π and $\pi\ o\ \alpha$ not equivalent, $\alpha^{**}(z)$ is different from z by [83]. Thus α^{**}does not fixes pointwise the center of A^{**} and, by Lemma (4.2.23), α is not approximately inner.

Proposition (4.2.24) shows that any simple nonelementary C^*-algebra contains a central sequence that is not strict-hypercentral. It is clear that the same conclusion holds for any C^*-algebra containing a simple nonelementary C^*-algebra as a direct summand. By Theorem 3.9 of [1], this class of C^*-algebras coincides with the class of C^*-algebras that have only inner derivations and do not have continuous trace. This concludes the proof of the following proposition:

Proposition(4.2.25)[278]: *If A is a C^*-algebra that does not have continuous trace and has only inner derivations, then A contains a central sequence that is not strict-hypercentral.*

In order to finish the proof of Theorem (4.2.1), it is enough to show that its conclusion holds for a C^*-algebra A containing a central sequence that is not strict-hypercentral.

Proposition(4.2.26)[278]: *If A is a C^*-algebra containing a central sequence that is not strict-hypercentral, then the approximately inner automorphisms of A are not classifiable by countable structures up to unitary equivalence.*

Proof: Fix a dense sequence $(a_n)_{n\in\mathbb{N}}$ in the unit ball of A. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a central sequence in A that is not strict-hypercentral. Thus there is a central sequence $(y_n)_{n\in\mathbb{N}}$ in A such that the sequence

$$([x_n, y_n])_{n\in\mathbb{N}}$$

does not converge strictly to 0. This implies that, for some positive contraction b in A, then the sequence

$$(b[x_n, y_n])_{n\in\mathbb{N}}$$

does not converge to 0 is norm. Without loss of generality we can assume that, for every $n \in \mathbb{N}, x_n$ and y_n are positive contractions. Observe that the sequence $(\exp(itx_k) - 1)_{n\in\mathbb{N}}$ is not strict-hypercentral for any $t \in (0,1)$. After passing to subsequences, we can assume that for some strictly positive real number ε, for every $t \in (0,1)$, every $s \in \left(\frac{1}{2}, 1\right)$, and every $n, k \in \mathbb{N}$ such that $k \leq n$:

(a) $\|[(a_k, exp(it.x_n))]\| < 2^{-n}$;

(b) $\|b[x_n, y_n]\| \geq \varepsilon$;

(c) $\|b[exp(isx_n), y_n]\| \geq \varepsilon$;

Define $\eta = \frac{\varepsilon}{20}$. After passing to a further subsequence, we can assume that, for every $t \in (0,1)$ and every $n, k \in \mathbb{N}$ such that $k \leq n$:

(a) $\|[exp(it.x_k), y_n]\| < 2^{-n}\eta$;

(b) $\|[y_k, exp(it.x_n)]\| < 2^{-n}\eta$;

(c) $\|[exp(it.x_k), exp(is.x_n)]\| < 2^{-n}\eta$.

It is not difficult to verify that, if $t \in (0,1)$ then the sequence

$$(\mathrm{Ad}(\exp(it_n x_n)))_{n\in\mathbb{N}}$$

is Cauchy in $\mathrm{Aut}(A)$. Denoting by $f(t)$its limit, one obtains a function

$$f: (0,1)^{\mathbb{N}} \to \overline{\mathrm{Inn(A)}}.$$

In the rest of the proof we will show that f satisfies the hypothesis of Criterion (4.2.6). Suppose that M is a Lipschitz constant for the function $t \mapsto exp(it)$ on $[0, 1]$. If $t, s \in (0,1)^{\mathbb{N}}$ and $n \in \mathbb{N}$ are such that $|t_k - s_k| < \varepsilon$ for $k \in \{1, 2, \dots, n\}$ then it is easy to see that

$$\|f(t)(a_k) - f(s)(a_k)\| \leq 2^{-n+1} + \varepsilon M$$

for $k \leq n$.This shows that the function f is continuous, particularly, Baire measurable. Moreover, if $t, s \in (0,1)^{\mathbb{N}}$are such that $s - t \in \ell^1$, then the sequence

$$\left(\exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1)\right)_{n\in\mathbb{N}}$$

is Cauchy in $U(A)$, and hence has a limit $u \in U(A)$. It is now readily verified that

$$f(t) = \mathrm{Ad}(u) \; o \; f(s)$$

and hence $f(t)$and $f(s)$are unitarily equivalent. Finally, suppose that C is a comeager subset of $(0,1)^{\mathbb{N}}$. Thus, there are $t, s \in C$ such that $|t_n - s_n| \in \left(\frac{1}{2}, 1\right)$for infinitely many $n \in \mathbb{N}$.We claim that $f(t)$and $f(s)$are not unitarily equivalent. Suppose by contradiction that this is not the case. Thus there is $u \in U(A)$such that

$$f(t) = \mathrm{Ad}(u) \text{ o } f(s).$$

This implies that the sequence

$$\left(u \exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1)\right)_{n\in\mathbb{N}}$$

in $U(A)$is central, i.e. asymptotically commutes (in norm) with any element of A. Fix now any ts $n_o \in \mathbb{N}$,such that $|t_{no} - s_{no}| \in \left(\frac{1}{2}, 1\right)$ and

$$\|b[y_n, u]\| < \eta$$

for $n \geq n_o$. Suppose that $n > n_o$. Using the fact that the elements $\exp(it_mx_m)$ and $\exp(it_kx_k)$ commute up to $5\eta^{-m}$ for $k, m \in \mathbb{N}$, one can show that

$$buy_{no}\, u \exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1)$$

is at distance at most 5η from

$$buy_{no} \exp(i(t_{no} - s_{no})x_{no})\exp(it_1x_1) \cdots \exp(\widehat{\imath t_{no}x_{no}})$$
$$\cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(\widehat{\imath s_{no}x_{no}}) \cdots \exp(-is_1x_1),$$

where $\exp(\widehat{\imath t_{no}x_{no}})$and $\exp(\widehat{\imath s_{no}x_{no}})$indicate omitted terms in the product. Similarly

$$bu \exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1)_{y_{no}}$$

is at distance at most 5η from

$$bu \exp(i(t_{no} - s_{no})x_{no})y_{no}\exp(it_1x_1) \cdots \exp(\widehat{\imath t_{no}x_{no}})$$
$$\cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(\widehat{\imath t_{no}x_{no}}) \cdots \exp(-is_1x_1)$$

Thus, the norm of the commutator of

$$u \exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1)$$

and y_0 is at least

$$\|b[\exp(i(t_{no} - s_{no})x_{no}), y_{no}]\| - 10\eta \geq \varepsilon - 10\eta \geq \frac{\varepsilon}{2}.$$

This contradicts the fact that the sequence

$$(u \exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1))_{n\in\mathbb{N}}$$

is central and concludes the proof.

If A is a C^*-algebra, then we denote by $A_0(A)$ the separable Banach space of inner derivations of A endowed with the norm $\|.\|\Delta_0(A)$ and by $\overline{\Delta_0(A)}$the closure of $\Delta_0(A)$ inside the space $\Delta(A)$ of derivations of A endowed with the operator norm. Suppose that $E_{\Delta(A)}$ is the Borel equivalence relation on $\overline{\Delta_0(A)}$ defined by

$$\delta_0 E_{\Delta(A)} \delta_1 \text{ iff } \delta_0 - \delta_1 \in \Delta_0(A).$$

Observe that $E_{\Delta(A)}$ is the orbit equivalence relation associated with the continuous action of the additive group of $\Delta_0(A)$on $\overline{\Delta_0(A)}$by translation.

Theorem(4.2.27)[278]: If A is a unital C^*-algebra, then the following statements are equivalent:

(i) $\Delta_0(A)$ *is closed in* $\Delta(A)$;

(ii) $E_{\Delta(A)}$ *is smooth;*

(iii) $E_{\Delta(A)}$ *is classifiable by countable structures;*

A has continuous trace.

The equivalence of (i) and (iv) follows from [42] together with the equivalence of (i) and (iii) in Theorem (4.2.2). The implication (i)⇒(ii) follows from [259]. Trivially (ii) is stronger than (iii). Finally observe that $\Delta_0(A)$and $\Delta_0(A)$satisfy the hypothesis of [301]. In fact, as discussed at the beginning, $\Delta_0(A)$endowed with the norm

$$\|\mathrm{ad}(ia)\|_{\Delta(\mathrm{A})} = \inf\left\{\|a \perp z\| \; z \in A' \cap A\right\}$$

is a separable Banach space. Moreover the inclusion map of $\Delta_0(A)$in $\overline{\Delta_0(A)} \subset \Delta(A)$ is bounded with norm at most (ii). Thus, if $\Delta_0(A)$is not closed in $\Delta(A)$then the continuous action of the additive group $\Delta_0(A)$on $\Delta_0(A)$by translation is turbulent. Hjorth's turbulence theorem recalled at the beginning concludes the proof of the implication (iii)⇒(i).

The implication (iii)⇒(i) of Theorem (4.2.1) does not hold in general. Remark 0.9 of [60] provides an example of a C^*-algebra A that has continuous trace such that the group $\mathrm{Inn}(A)$of inner automorphisms of A is not closed inside $\mathrm{Aut}(A)$. This implies that the automorphisms of A are not concretely classifiable up to unitary equivalence. It would be interesting to know if the automorphisms of A are at least classifiable by countable structures up to unitary equivalence.

Corollary(4.2.28)[370]: Fix a strictly positive real number η. For every $\varepsilon > 0$ there is $\delta > 0$ such that for every C^*-algebra A and every pair of positive contractions $x_r^2, a^2 + \epsilon$ of A such that $\|a^2 + \epsilon\| \geq \eta$, if

$$\sum_r \|(exp(i\,x_r^2) - \mu_r^2)(a^2 + \epsilon)\| \leq \delta$$

for some $\mu_r^2 \in \mathbb{C}$ then

$$\sum_r \|(x_r^2 - \lambda^2)(a^2 + \epsilon)\| \leq \varepsilon$$

for some $\lambda^2 \in \mathbb{C}$.

Proof: Fix $\varepsilon > 0$. Let L be the principal branch of the logarithm. Since L is an analytic function on the open disc of radius 1 centered in 1, there is a polynomial

$$p(Z) = \rho_0, \rho_1 Z + \cdots + \rho_n Z^n$$

Such that

$$\sum_r |p(z_r^2) - L(z_r^2)| \leq \frac{\varepsilon}{2}$$

for every $z_r^2 \in \mathbb{C}$ such that $\sum_r |z_r^2 - 1| \leq \exp(i)$. In particular for every $t \in [0,1]$

$$\left|p\big(\exp(it)\big) - \mathrm{t}\right| = \left|p\big(\exp(it)\big) - \mathrm{L}\big(\exp(it)\big)\right| \leq \frac{\varepsilon}{2}.$$

If $\mu_r^2 \in \mathbb{C}$ is such that $|\mu_r^2| \leq \frac{2}{\eta}$,define ${}_{\rho_{\mu_r^2}}(z_r^2)$ to be the polynomial in Z obtained from $p(Z)$ by replacing the indeterminate Z by $Z + \mu_r^2$. Observe that the j-th coefficient of ${}_{\rho_{\mu_r^2}}(z_r^2)$ is

$$\sum_r \rho_j^{\mu_r^2} = \sum_{i=j}^{n} \sum_r |\rho_i| \binom{i}{j} \mu_r^{2(j-i)}$$
for $0 \le j \le n$. Finally define
$$C = \sum_{1\le j \le i \le n} |\rho_i| \binom{i}{j} \left(\frac{3}{\eta}\right)^{j-i} \left(\frac{2}{\eta}\right)^{j-1}$$
and
$$\delta = min\left\{\frac{\varepsilon}{2C}, 1\right\}.$$
Suppose that A is a C^*-algebra and $x_r^2, a^2+\epsilon \in A$ are positive contractions such that $\|a^2+\epsilon\| \ge \eta$ and, for some $\mu_r^2 \in \mathbb{C}$,
$$\sum_r \|(\exp(ix_r^2) - \mu_r^2)(a^2+\epsilon)\| \le \delta.$$
Thus,
$$\sum_r |\mu_r^2| \le \frac{2}{\eta}.$$
Moreover
$$\sum_r \left\|\left(x_r^2 - \rho_0^{\mu_r^2}\right)(a^2+\epsilon)\right\| = \sum_r \left\|\left(p\left(\exp(ix_r^2)\right) - \rho_0^{\mu_r^2}\right)(a^2+\epsilon)\right\| + \frac{\varepsilon}{2}$$
$$= \sum_r \left\|\left(\sum_{j=1}^{n} \rho_j^{\mu_r^2} (\exp(ix_r^2) - \mu_r^2)^j\right)(a^2+\epsilon)\right\| + \frac{\varepsilon}{2}$$
$$\le \sum_{j=1}^{n} \sum_r \left|\rho_j^{\mu_r^2}\right| \|\exp(ix_r^2) - \mu_r^2\|^{j-1}\, \delta + \frac{\varepsilon}{2}$$
$$\le \sum_{j=1}^{n} \sum_{i=j}^{n} |\rho_i| \binom{i}{j} \left(\frac{2}{\eta}\right)^{j-i} \left(\frac{3}{\eta}\right)^{j-1} \delta + \frac{\varepsilon}{2} \le C\delta + \frac{\varepsilon}{2} \le \varepsilon.$$
This concludes the proof.

Corollary(4.2.29)[370]: If $(x_n^m)_{n\in\mathbb{N}}$ is a strict-hypercentral sequence in A and α_m is an approximately inner automorphism of A, then $(\alpha_m(x_n^m) - x_n^m)_{n\in\mathbb{N}}$ converges strictly to 0.

Proof: The same proof of Kaplansky's density theorem [83] shows that the unit ball of A is strictly dense in the unit ball of $M(A)$; see [291]. (The strict topology on the multiplier algebra of A has been defined.) It follows that, if $\varepsilon > 0$ and a is an element of A, then there is a finite subset F of the unit ball of A, a positive real number δ, and a natural number n_0 such that, for every $n \ge n_0$ and every y in the unit ball $M(A)$such that $\|[y, z^m]\| \le \varepsilon$ for every $z^m \in F$,
$$\sum_m \max\{\|a(x_n^m y - y x_n^m)\|, \|(x_n^m y - y x_n^m)a\|\} \le \varepsilon.$$
Consider the open neighbourhood
$$U = \{\alpha_m \in \mathrm{Aut}(A) \mid \|\alpha_m(x^m) - x^m\| < \delta \text{ for every } x^m \in F\}$$
of id_A in $\mathrm{Aut}(A)$. Observe that if $\beta_m \in U$ is inner, then for every $n \ge n_n$

$$\sum_m \|(\beta_m(x_n^m) - x_n^m)a\| \leq \varepsilon$$

and

$$\sum_m \|a(\beta_m(x_n^m) - x_n^m)\| \leq \varepsilon.$$

Approximating with inner automorphisms, one can see that the same is true if $\beta_m \in U$ is just approximately inner. Since α_m is approximately inner, there is a unitary multiplier u of A and an approximately inner automorphism β_m of A in U such that

$$\sum_m \alpha_m = \sum_m \beta_m \, o \, \mathrm{Ad}(u).$$

Consider a natural number $n_1 \geq n_0$ such that, for $n \geq n_1$,

$$\sum_m \|\beta_m^{-1}(a)[x_n^m, u\,]\| \leq \varepsilon$$

and

$$\sum_m \|[x_n^m, u^*]\beta_m^{-1}(a)\| \leq \varepsilon.$$

It follows that, if $n \geq n_1$,

$$\sum_m \|a(\alpha_m(x_n^m) - x_n^m)\| \leq \sum_m \left\|a\beta_m\big(\mathrm{Ad}(u)(x_n^m)\big) - x_n^m\right\| + \sum_m \|\beta_m(z_n^m) - x_n^m\|$$

$$\leq \sum_m \|\beta_m^{-1}(a)(ux_n^m u^* - z_n^m)\| + \varepsilon = \sum_m \|\beta_m^{-1}(a)[x_n^m, u\,]\| + \varepsilon \leq 2\varepsilon$$

and, analogously,

$$\sum_m \|(\alpha_m(x_n^m) - x_n^m)a\| \leq 2\varepsilon.$$

Since ε was arbitrary, this concludes the proof of the fact that

$$(a(z_n^m) - x_n^m)_{n\in\mathbb{N}}$$

converges strictly to 0.

Chapter 5
Model Theory and Countable Chain Condition with Saturation

We show a purely model-theoretic result to the effect that the theory of a separable metric structure is stable if and only if all of its ultrapowers associated with nonprincipal ultrafilters on ℕ are isomorphic even when the Continuum Hypothesis fails. We show independence from ZFC of the statement that this condition is preserved under the tensor products of C*-algebras. We also characterize elementary equivalence of the algebras $C(X)$ in terms of $CL(X)$ when X is 0-dimensional, and show that elementary equivalence of the generalized Calkin algebras of densities $\aleph_\alpha$ and $\aleph_\beta$ implies elementary equivalence of the ordinals α and β.

Section (5.1): Operator Algebras

We study operator algebras using a slightly modified version of the model theory for metric structures. This is a logical framework whose semantics are well-suited for the approximative conditions of analysis; as a consequence it plays the same role for analytic ultrapowers as first order model theory plays for classical (set theoretic) ultrapowers. We show that the continuum hypothesis (CH) implies that all ultrapowers of a separable metric structure are isomorphic, but under the negation of CH this happens if and only if its theory is stable. Stability is defined in logical terms (the space of φ-types over a separable model is itself separable with a suitable topology), but it can be characterized as follows: a theory is not stable if and only if one can define arbitrarily long finite funiformly well-separated" totally ordered sets in any model, a condition called the order property. Provided that the class of models under consideration (e.g., II factors) is defined by a theory - not always obvious or even true - this brings the main question back into the arena of operator algebras. To deduce the existence of nonisomorphic ultrapowers under the negation of CH, one needs to establish the order property by defining appropriate posets. We proved in [120] that all infinite-dimensional C^*-algebras and II_1 factors have the order property, while tracial von. Neumann algebras of type I do not. We will use the logic developed here to obtain new results about isomorphisms and embeddings between II_1factors and their ultrapowers.

We now review some facts and terminology for operator algebraic ultrapowers from [120].

A von Neumann algebra M is tracial if it is equipped with a faithful normal tracial state tr.

A finite factor has a unique tracial state which is automatically normal. The metric induced by the ℓ^2-norm, $\|a\|_2 = \sqrt{tr(a^*a)}$, is not complete on M, but it is complete on the (operator norm) unit ball of M. The completion of M with respect to this metric is isomorphic to a Hilbert space (see, e.g., [100] or [315]).

The algebra of all sequences in M bounded in the operator norm is denoted by $\ell^\infty(M)$. If $\mathcal{U}$ is an ultrafilter on ℕ then

$$c_u = \left\{\underset{a}{\rightarrow} \in \ell^\infty(M) : \lim_{i\to u} \|a_i\|_2 = 0\right\}$$

is a norm-closed two-sided ideal in $\ell^\infty(M)$, and the tracial ultrapower M^u(also denoted by$\prod_u M$) is de_ned to be the quotient $\ell^\infty(M) / c_u$. It is well-known that M^u is tracial, and a factor if and only if M is| see, e.g., [100] or [321]; this also follows from axiomatizability and Los's theorem (Proposition (5.1.8) and the remark afterwards).

Elements of M^u will either be denoted by boldface Roman letters such as a or represented by sequences in $\ell^\infty(M)$. Identifying a tracial von Neumann algebra M with its diagonal image in M^u, we will also work with the relative commutant of M in its ultrapower,

$$M' \cap M^u = \{ b : (\forall a \in M) ab = ba\}$$

Tracial ultrapowers were first constructed in the 1950s and became standard tools after the groundbreaking of McDuff ([317]) and Connes ([156]). The properties of an ultrapower are the approximate properties of the initial object; see [320].

In defining ultrapowers for C^*-algebras (resp. groups with bi-invariant metric), c_u is taken to be the sequences that converge to zero in the operator norm (resp. converge to the identity in the metric ([318])). All these constructions are special cases of the ultrapower/ultraproduct of metric structures (see [303] or [311]).

The purpose is to introduce a logic which has some features geared to the treatment of C^*-algebras and von Neumann algebras. In a treatment of such structures in bounded continuous logic (see [305]), it is typical to consider different sorts of balls of increasing radius. The logic presented here is entirely equivalent to that formulation but allows us to introduce function symbols like $+$ and . without treating them as infinitely many different functions mapping between sorts. This distinction is somewhat cosmetic but the treatment of terms in this logic highlights an issue that is common to both this logic and the multisorted version. Details are given below but to make clear what is at stake, suppose we are considering a normed linear space and we wish to assert that the unit ball is convex. The operation $+$ when restricted to the unit ball would most naturally map to the ball of radius 2. Scalar multiplication by $1/2$ maps the ball of radius 2 into the unit ball and so a natural way to set things up would be to have the term $(x + y) / 2$ send the unit ball to itself and so the syntax guarantees that the unit ball is convex. If on the other hand, the scalar $1/2$ on the ball of radius 2 was said to have range that same ball (a logical possibility), then $(x + y) / 2$ syntactically would only map the unit ball to the ball of radius 2 and we would need to have an axiom that said that this term in fact has range in the unit ball. Issues of the axiomatizability of the classes of structures we are dealing with are bound up with the choice of range of terms in our language and are highlighted below.

A language consists of

(a) Sorts, S, and for each sort $S \in S$, a , set of domains D_S meant to be domains of quantification, and a privileged relation symbol d_S intended to be a metric. Each sort comes with a distinct set of variables.

(b) Sorted functions, $f: S_1 \times ... \times S_n \to S$ together with, for every choice of domains $D_i \in D_{S_i}$, a $D_{\overline{D}}^f \in D_S$ and for each i, a uniform continuity modulus $\delta_i^{\overline{D},\mathcal{T}}$, $i.e.,a$ real-valued function on $\mathbb{R}$, where $\overline{D} = \langle D_1, ... D_n\rangle$

(c) Sorted relations R on $S_1 \times ... \times S_n$ there is a number such that for every choice of domains $\overline{D}$ as above, as well as uniform continuity moduli dependent on i and $\overline{D}$.

(d) Terms are formed by the usual composition of function symbols and variables. They inherit codomains and series of uniform continuity moduli from this composition.

A structure $\mathcal{M}$ assigns to each sort $S \in S, M(S)$, a metric space where d_S is interpreted as the metric. For each $D \in D_S$, $M(D)$ is a subset of $M(S)$ complete with respect to d_S. The collection $\{M(D) : D \in D_S\}$covers $M(S)$.

Terms t are interpreted as functions on a structure in the usual manner. If t^M is the interpretation of t and $\overline{D}$ is a choice of domains from the relevant sorts then $t^M: M(\overline{D}) \to$

$M(D_{\bar{D}}^{t})$ and t^M is uniformly continuous as specified by the $\delta^{\bar{D},t}\,'s$ when restricted to $M(\bar{D})$.
This means for instance that for every $\epsilon > 0$, if $a, b \in M(D_1)$ and $c_i \in M(D_i)$ for $i = 2, \dots, n$ then $d_S(a,b) < \delta_1^{\bar{D},t}(\epsilon)$ implies $d_{S'}\,({}_t(a,\bar{c}), t(b,\bar{c})) \le \epsilon$, where S is the sort associated to D_1 is the sort associated with the range of t.
Sorted relations are maps $R^M\colon S_1 \times \dots \times S_n \to \mathbb{R}$. They are handled similarly to sorted functions; uniform continuity is as above when restricted to the appropriate domains and a relation R is bounded in absolute value by $N_{\bar{D}}^{R}$ when restricted to $M(\bar{D})$.
We will think of a C^*-algebra A as a one-sorted structure with sort U for the algebra itself. The domains for U are D_n for every $n \in \mathbb{N}$ and are interpreted as all $x \in A$ with $\|x\| \le n$. The metric on U is

(a) The constant 0 which will be in D_1. Note it is a requirement of the language to say this.
(b) For every $\lambda \in \mathbb{C}$ a unary function symbol also denoted λ to be interpreted as scalar multiplication. For simplicity we shall write $\lambda\, x$ instead of $\lambda\,(x)$.
(c) A unary function symbol $*$ for involution.
(d) Binary function symbols $+$ and .

Prescribing the uniform continuity moduli is straightforward.
If A is a C^*-algebra then there is a model, $\mathcal{M}(A)$, in $\mathcal{L}_{C^*}$associated to it which is essentially A itself endowed with the domains D_{n} interpreted as the operator norm n-ball.
Tracial von Neumann algebras will be treated as a onesorted structure with domains D_n which as in the example of C^*-algebras will be interpreted as the operator norm n-ball. The metric d will be the metric arising from the ℓ^2 norm coming from the trace.
The functions in the language are, in addition to functions,

(a) The constant 1 in D_1.
(b) Two unary relation symbols tr^{r} and tr^{i} for the real and imaginary parts of the trace function. We will often just write tr and assume that the expression can be decomposed into the real and imaginary parts.

Again, this describes a language $\mathcal{L}_{Tr}$ once we add the requirements about bounds on the range and uniform continuity.
If N is a tracial von Neumann algebra then there is a model, $M(N)$, in $\mathcal{L}_{Tr}$ associated to it which is essentially N itself with the domains interpreted as above.
The syntax for logic of unitary groups is simpler than that of tracial von Neumann algebras or C^*-algebras. In this case the metric is bounded and therefore we can have one domain are equal to the universe U. We have function symbols for the identity, inverse and the group operation. Since in this case our logic reduces to the standard logic of metric structures as introduced in [303] we omit the straightforward details and continue this practice of suppressing the details for unitary groups throughout.
(a) Formulas:
(i) If R is a relation and $t_1, \dots, t_n$ are terms then $R(t_1, \dots, t_n)$ is a basic formula.
(ii) If $f\,:\,\mathbb{R}^n \to \mathbb{R}$ is continuous and $\varphi_1, \dots, \varphi_n$ are formulas, then $R(\varphi_1, \dots, \varphi_n)$is a formula.
If $D \in \mathcal{D}_S$ and φ is a formula then both $\sup_{x\in D} \varphi$ and $\mathrm{in}f_{x\in D}\varphi$ are formulas.
(b) Formulas are interpreted in the obvious manner in structures. The boundedness of relations when restricted to domains is essential to guarantee that the sups and infs exist when interpreted. For a fixed formula φ and real number r, the expressions $\varphi \le r$ and

$r \leq \varphi$ are called conditions and are either true or false in a given interpretation in a structure.

In the above definition it was taken for granted that we have an infinite supply of distinct variables appearing in terms. We shall need to introduce a set of new constant symbols C. Each $c \in C$ is assigned a sort $S(c)$ and a domain.
In the expanded language $\mathcal{L}_C$ both variables and constant symbols from C appear in terms. Formulas and sentences in $\mathcal{L}_C$ are defined as above. Note that, since the elements of C are not variables, we do not allow quantification over them.
A sentence is a formula with no free variables. If $\varphi$is a sentence and $\mathcal{M}$ is a structure then the result of interpreting φ in $\mathcal{M}$ is a real number, $\varphi^{\mathcal{M}}$. The function which assigns these numbers to sentences is the theory of , denoted by Th$(\mathcal{M})$. Because we allow all continuous functions as connectives, in particular the functions $|x - \lambda|$, the theory of a model $\mathcal{M}$ is uniquely determined by its zero-set$\{\varphi : \varphi^{\mathcal{M}} = 0\}$. We shall therefore adopt the convention that a set of sentences T is a theory and say that $\mathcal{M}$ is a model of $T, \mathcal{M} = T,$ if $\varphi^{\mathcal{M}} = 0$ for all $\varphi \in T$.

The following is proved by induction on the complexity of the definition of ψ .

Lemma(5.1.1)[302]: Suppose $\mathcal{M}$ is a model and $\psi(\bar{x})$ is a formula, possibly with parameters from M. For every choice of $\bar{D}$ sequence of domains consistent with the sorts of the variables, ψ^{M} is a uniformly continuous function on $M(\bar{D})$ into a compact subset of $\mathbb{R}$.

If $\Theta: \mathcal{M} \to \mathcal{N}$ is an isomorphism then $\psi^{\mathcal{M}} = \psi^{\mathcal{N}} \circ \Theta$.

Two models $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent if $Th(\mathcal{M}) = Th(\mathcal{N}). A$ map $\Theta: \mathcal{M} \to \mathcal{N}$ is an elementary embedding if for all formulas ψ with parameters in M, we have $\psi^{\mathcal{M}} = \psi^{\mathcal{N}} \circ \Theta$.

If $\mathcal{M}$ is a submodel of $\mathcal{N}$ and the identity map from $\mathcal{M}$ into $\mathcal{N}$ is elementary then we say that $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$. It is not difficult to see that every elementary embedding is an isomorphism onto its image,1but not vice versa.

Definition(5.1.2)[302]: A category C is axiomatizable if there is a language $\mathcal{L}$ (as above), theory T in $\mathcal{L}$, and a collection of conditions $\sum$ such that C is equivalent to the category of models of T with morphisms given by maps that preserve$\sum$.

The reason for being a little fussy about axiomatizability is that in the cases we wish to consider, the models have more (albeit artificial) `structure' than the underlying algebra. The language of the model will contain operation symbols for all the algebra operations (such as $+$,.and $*$) and possibly some distinguished constant symbols (such as the unit) and predicates (e.g., a distinguished state on a C^*-algebra). It will also contain domains that are not part of the algebra's structure.

In particular then, when we say that we have axiomatized a class of algebras C, we will mean that there is a first order continuous theory T and specification of morphisms such that

(a) for any $A \in C$, there is a model $M(A)$ of T determined up to isomorphism;

(b) for any model M of T there is $A \in C$ such that M is isomorphic to $M(A)$;

(c) if $A;\ B \in C$ then there is a bijection between Hom$(A;\ B)$ and Hom $(M(A);\ M(B))$.

Proving that a category is axiomatizable frequently involves somewhat tedious syntactical considerations. However, once this is proved we can apply a variety of model-theoretic tools to study this category. We can immediately conclude that the category is closed under taking ultraproducts|a nontrivial theorem in the case of tracial von Neumann

algebras. From here it also follows that some natural categories of operator algebras are not axiomatizable (see Proposition (5.1.28)).
We continue the discussion of model theory of C^*-algebras started. First we introduce two notational shortcuts. If one wants to write down axioms to express that $t = \sigma$ for terms t and σ then one can write

$$\varphi_{\bar{D}}: \operatorname{Sup}_{\bar{a}\in\bar{D}} d_U\big(t(\bar{a}), \sigma(\bar{a})\big)$$

where $\bar{D}$ ranges over all possible choices of domains. Note that this is typically an infinite set of axioms. Remember that for a model to satisfy $\varphi_{\bar{D}}$, this sentence would evaluate to 0 in that model. If this sentence evaluates to 0 for all choices of $\bar{D}$then clearly $t = \sigma$ in that model.
If one wants to write down axioms to express that $\varphi \geq \psi$for formulas $\varphi$and ψ then one can write

$$\sup_{\bar{a}\in\bar{D}} \max\Big(0, \big(\psi(\bar{a}) - \varphi(\bar{a})\big)\Big)$$

where $\bar{D}$ ranges over all possible choices of domains. Again, we will get the required inequality if all these sentences evaluate to 0 in a model.
Using the above conventions, we are taking the universal closures of the following formulas ,where $x;\ y;\ z;\ a;\ b$, range over the algebra and λ, μ range over the complex numbers.
Here are some sentences that evaluate to zero in a C^*-algebra A. The first item guarantees that we have a $\mathbb{C}$-vector space.
(i) $x + (y + z) = (x + y) + z, x + 0 = x, x + (x) = 0$ (where$-x$ is the scalar -1 acting on x), $x + y = y + x, \lambda(\mu x) = (\lambda\mu)x, \lambda(x + y) = \lambda x + \lambda y, (\lambda + \mu)x = \lambda x + \mu x$.
(ii) $1x = x, x(yz) = (xy)z, \lambda(xy) = (\lambda x)y = x(\lambda y), x(y + z) = xy + xz$; now we have $a\ C^* -$ algebra.
(iii) $(x^*)^* = x, (x + y)^* = x^* + y^*, (\lambda x)^* = \bar{\lambda} x^*$.
(iv) $(xy)^* = y^*x^*$.
(v) $d_U(x;\ y) = d_U(x\ \ y, 0)$; we will write $\|x\|$ for $d_U(x, 0)$.
(vi) $\|xy\| \leq \|x\|\|y\|$.
(vii) $\|\lambda x\| = |\lambda|\|x\|$.
(viii) ($C^* -$ equality) $\|xx^*\| = \|x\|^2$.
(ix) $\sup_{a\in D_1}\|a\| \leq 1$.
One issue here is that these axioms are too weak to guarantee that D_1 is the operator norm unit ball. To get around this we expand the language of C^*-algebras to include a functionsymbol t_{p} for every $*$-polynomial p in one variable. The symbolt_{p}will have the same uniform continuity modulus as p. In order to determine the proper codomains, for every n, let m be the least integer greater than or equal to $\max\{\|p(a)\| : a \in M, M \in C$ and $\|a\| \leq n\}$ where C is the class of C^*-algebras. We will require $t_p: D_n \to D_m$ and we will add the universally quantified axioms
(x) $t_p(x) = p(x)$ for all polynomials p. This will force the polynomial p to behave well with respect to where its range lands. To see the effect of these axioms, we do a small calculation.
Suppose that $\mathcal{M}$ is a structure that satisfies axioms 1 through 9 above. Suppose $a \in M, \|a\| \leq 1$ and $a \in D_n(\mathcal{M})$. Define

$$t_n(x) = \begin{cases} 1 & 0 \le x \le 1 \\ \frac{1}{\sqrt{x}_6} & 1 < x \le n \end{cases}$$

and consider $f(u) = ut_n(u^* u)$. If we want to compute the norm of $f(u)$ for $\|u\| \le n$, we see that $\|f(u)\|^2 = \|t_n(u^*u)u^*ut_n(u^*u)\| = \|g(u^*u)\|$ where $g(x) = xt_n^2(x)$. Since

$$g(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 < x \le n \end{cases}$$

we obtain that the norm of $f(u)$ is at most 1 when $\|u\| \le n$.

Now fix polynomials $p_k(x)$ which tend to $t_n(x)$ from below on the interval $[0, n]$. By doinga calculation similar to the one above, the $*$-polynomial $q_k = up_k(u^*u)$ sends operators of norm $\le n$ to operators of norm ≤ 1. This means that T_{q_k} sends elements of D_n to elements of D_1 by the specification of our language for C^*-algebras. Moreover, $ap_k(a^*a)$ tends to a as k tends to infinity. Since $D_1(\mathcal{M})$ is complete, we obtain that $a \in D_1(\mathcal{M})$.

Proposition(5.1.3)[302]: The class of C^*-algebras is axiomatizable by theory T_{C^*} consisting of axioms (i)-(x).

Proof: It is clear that for a C^*-algebra A the model $M(A)$ as defined in satisfies T_{C^*}.

Conversely, if a model M of $\mathcal{L}_{C^*}$ satisfies T_{C^*} then the algebra A_M obtained from M by forgetting the domains is a C^*-algebra by Gel'fand-Naimark.

To see that this provides an equivalence of categories, we only need to show that $M(A_M) \cong M$. To see this, we must show that the domains on M are determined by A_M. Since multiplication by a scalar r provides a bijection between the operator norm unit ball and the ball of radius r, it suffices to show that the operator norm unit ball and those elements of $D_1(M)$ coincide. By axiom 9, we have that the latter is contained in the former. The other direction is just the calculation we did immediately before the Proposition.

We continue our discussion of model theory of tracial von Neumann algebras. Axioms for tracial von Neumann algebras and II_1 factors appear in the context of bounded continuous logic in [304]; those axioms are restricted to axiomatizing the norm one unit ball. We feel in this context axiomatizing von Neumann algebras in the logic described in the previous makes the axioms more natural. Here are some sentences that evaluate to zero in a tracial von Neumann algebra N:

(xi) All axioms (i)-(v) plus $1x = x = x1$ for the constant 1 of N. In case of (v) we will write $\|x\|_2$ for $d_U(x; 0)$.

(xii) $tr(x + y) = tr(x) + tr(y)$

(xiii) $tr(x^*) = \overline{tr(x)}, tr(\lambda x) = \lambda\, tr(x), tr(xy) = tr(yx), tr(1) = 1,$

(xiv) $tr(x^*x) = \|x\|_2^2$.

Any model of these axioms will be a tracial $*$-algebra. The remaining axiom will guarantee that the relationship between the domains and the 2-norm is correct.

(xv) For every $n, m \in \mathbb{N}$,

$$\sup_{a \in D_n} \sup_{x \in D_m} \max\{0, \|ax\|_2 - n\|x\|_2\}$$

In addition to these axioms, we also introduce terms t_p for all unary $*$-polynomials p as discussed above for C^*-algebras.

Proposition(5.1.4)[302]: The class of tracial von Neumann algebras is axiomatizable by theory T_{T_r} consisting of axioms (x)-(xv).

Proof: It is clear that for a tracial von Neumann algebra N the model $\mathcal{M}(N)$ as defined in §2.3.2 satisfies T_{T_r}. Assume $\mathcal{M}$ satisfies T_{T_r}. To see that in the sort U we have a tracial von Neumann algebra suppose A is the underlying set for U in $\mathcal{M}$. Then A is a complex pre-Hilbert space with inner product given by $tr(y^*x)$. Left multiplication by $a \in A$ is a linear operator on A and axiom (xv) guarantees that a is bounded. The operation$*$ is the adjoint because for all x and y we have $\langle ax, y\rangle = tr(y^*ax) = tr((a^*y)^*x) = \langle x, a^*y\rangle$. Thus A is faithfully represented as a $*$-algebra of Hilbert space operators. We know that $D_n(A)$ is complete with respect to the 2-norm for all n and the 2-norm induces the strong operator topology on A in this representation; it follows from the Kaplansky density theorem that A is a tracial von Neumann algebra.

As in the case of C^*-algebras above, to show that we have an equivalence of categories, it will suffice to show that if $\mathcal{M}$ is a model of the T_{T_r} then $D_1(A)$ is given by the operator norm unit ball on A. Axiom (xv) guarantees that $a \in D_1(A)$ then $\|a\| \leq 1$ and the functional calculus argument from the proof of Proposition(5.1.3) shows $D_1(A)$equals the operator norm unit ball.

For a in a tracial von Neumann algebras define the following:

$$\xi(a) = \sqrt{\|a\|_2^2\ tr^2(a)},$$

$$\eta(a) = \sup_{b\in D_1} \|ab - ba\|_2$$

1Since ξ and η are interpretations of terms in the language of tracial von Neumann algebras, the following is a sentence of this language.

(xvi) $\sup_{a\in D_1} \max\{0, (\xi(a)\ \eta(a))\}$.

Also consider the axiom

(xvii) $\inf_{a\in D_1} (\|aa^*\ (aa^*)^2\| + |tr(aa^*) - 1/\pi|)$.

Proposition(5.1.5)[302]: (i) The class of tracial von Neumann factors is axiomatizable by the theory consisting of axioms (x)-(xvi).

(ii) The class of II_1factors is axiomatizable by the theory T_{II_1} consisting of axioms (x)-(xvii).

Proof: For (i), by Proposition(5.1.4), it suffices to prove that if M is a tracial von Neumann algebra then axiom (xvi) holds in M if and only if M is a factor. If it is not a factor, let p be a nontrivial central projection. Then $\xi(p) = \sqrt{tr(p) - tr(p)^2} > 0$ but $\eta(p) = 0$, therefore (xvi) fails in M. If it is a factor, the inequality $\eta(a) \geq \xi(a)$ follows from [120].

For (ii) we need to show that axiom (xvii) holds in a tracial factor M if and only if M is type II_1. When M is type II_1, (xvii) is satisfied by taking a to be a projection of trace $1/\pi$. On the other hand, a tracial factor M not of type $\mathrm{II}_1$is some matrix factor $\mathbb{M}_k$. If $\mathbb{M}_k$ were to satisfy (xvii), by compactness of the unit ball there would be $a \in \mathbb{M}_k$ satisfying $\|(aa^*) - (aa^*)^2\|_2 = 0$ and $|\mathrm{tr}(aa^*)\ \ 1/\pi| = 0$. Thus $aa^* \in \mathbb{M}_k$would be a projection of trace $1/\pi$, which is impossible. (Of course this argument still works if $1/\pi$ is replaced with any irrational number in (0, 1).)

We introduce variants of some of the standard model-theoretic tools for the logic described. Assume $\mathcal{M}_i$, for $i \in I$, are models of the same language and $\mathcal{U}$ is an ultrafilter on I. The ultra product $\prod_{\mathcal{U}} \mathcal{M}_i$ is a model of the same language defined as follows.

In a model $\mathcal{M}$, we write S^{M} and D^{M} for the interpretations S and D in $\mathcal{M}$. For each sort $S \in S$, let

$$X_S = \left\{ \bar{a} \prod\nolimits_{i \in I} S^{\mathcal{M}_i} : \text{for some } D \in \mathcal{D}_S, \{i \in I : a_i\, D^{\mathcal{M}_i}\} \in \mathcal{U} \right\}.$$

For $\bar{a}$ and $\bar{b}$ in X_S, $d'_S(\bar{a}, \bar{b}) = \lim_{i \to \mathcal{U}} d_S^{\mathcal{M}_i}\ (a_i, b_i)$ defnes a pseudo-metric on X_S . Let $S^{\mathcal{M}'}$be the quotient space of X_S with respect to the equivalence $\bar{a} \sim \bar{b}$iff$d'_S(\bar{a}, \bar{b}) = 0$ and Let d_S be the associated metric. For $D \in \mathcal{D}_S$, let $S^{\mathcal{M}'}$ be the quotient of

$$\{\bar{a} \in X_S : \{i \in I : a_i \in D^{\mathcal{M}_i}\} \in \mathcal{U}\}.$$

All the functions and predicates are interpreted in the natural way. Their restrictions to each $\bar{D}$ are uniformly continuous and respect the corresponding uniform continuity moduli. If $\mathcal{M}_i = \mathcal{M}$ for all i then we call the ultraproduct an ultrapower and denote it by $\mathcal{M}^{\mathcal{U}}$.
The generalized ultraproduct construction' as introduced in [311] reduces to the model-theoretic ultraproduct in the case of both tracial von Neumann algebras and C^*-algebras.

We record a straightforward consequence of the definitions and the axiomatizability, that the functors corresponding to taking the ultrapower and defining a model commute. The ultrapowers of C^*-algebras and tracial von Neumann algebras are defined in the usual way.

Proposition(5.1.6)[302]: If A is a C^*-algebra or a tracial von Neumann algebra and $\mathcal{U}$ is an ultrafilter then $\mathcal{M}(A^{\mathcal{U}}) = \mathcal{M}(A)^{\mathcal{U}}$.

Corollary(5.1.7)[302]: A C^* -algebra(or a tracial von Neumann algebra) A has nonisomorphic ultrapowers if and only if the model $M(A)$ has nonisomorphic ultrapowers.

Proof: This is immediate by Proposition(5.1.3), Proposition(5.1.4) and Proposition (5.1.6).
It is worth remarking that although the proof of Proposition(5.1.6) is straightforward, this relies on a judicious choice of domains of quantification. In general, it is not true that if one defines domains for a metric structure then the domains have the intended or standard interpretation in the ultraproduct. Von Neumann algebras themselves are a case in point. If we had defined our domains so that D_n were those operators with l_2-norm less than or equal to n then there would be several problems. The most glaring is that these domains are not complete; even if one persevered to an ultraproduct, the resulting object would contain unbounded operators.

Ward Henson has pointed out to us that this same problem with domains manifests itself in pointed ultrametric spaces. If one defines domains as closed balls of radius n about the base point, there is no reason to expect that the domains in an ultraproduct will also be closed balls. This unwanted phenomenon can be avoided by imposing a geodesic-type condition on the underlying metric; see for instance [307].

The following is Los's theorem, also known as the Fundamental Theorem of ultraproducts (see [303]). It is proved by chasing the definitions.

Proposition(5.1.8)[302]: Let $\mathcal{M}_i$, $i \in \mathbb{N}$, be a sequence of models of language $\mathcal{L}, \mathcal{U}$ be an ultra filter on $\mathbb{N}$ and $\mathcal{N} = \prod_{\mathcal{U}} M_i$

(i) If ϕ is an $\mathcal{L}$ -sentence then $\phi^{\mathcal{N}} = \lim_{i \to \mathcal{U}} \phi^{\mathcal{M}_i}$:

(ii) If ϕ is an $\mathcal{L}$-formula then $\phi^{\mathcal{N}}(a) = \lim_{i \to \mathcal{U}} \phi^{\mathcal{M}_i}(a_i)$ where$(a_i : i \in \mathbb{N})$ is a representing sequence of a.

(iii) The diagonal embedding of a model $\mathcal{M}$ in to $\mathcal{M}^{\mathcal{U}}$ elementary.

Together with the axiomatizability (Propositions(5.1.3) and(5.1.4)) and Proposition(5.1.6), this implies the well-known fact that the ultraproduct of C^*-algebras (tracial von Neumann algebras, II_1 factors, respectively) is a C^*-algebra (tracial von Neumann algebra, II_1 factor, respectively).

In the setting of tracial von Neumann algebras, we have that for any formula $\phi(x_1,..,x_n)$with variables from the algebra sort there is a uniform continuity modulus δ such that for every tracial von Neumann algebra $\mathcal{M}, \phi$ defines a function g on the operator norm unit ball of $\mathcal{M}$ which is uniformly continuous with respect to δ and naturally extends to the operator norm unit ball of any ultrapower of $\mathcal{M}$.

In [120] we dealt with functions g satisfying the properties in the previous paragraph and used them to define a linear ordering showing that some ultrapowers and relative commutants are nonisomorphic. Using model theory, we can interpret this in a more general context and instead of `tracial von Neumann algebra' consider g defined with respect to any axiomatizable class of operator algebras. Clearly, Lemma(5.1.1) and Proposition(5.1.8) together imply the following, used in the proof of Theorem(5.1.27).

Corollary (5.1.9)[302]: If$\psi$is an n-ary formula, then the function g defined to be the interpretation of ψ on a tracial von Neumann algebra M satisfies the following [120]:

(G1) g defines a uniformly continuous function on the g -th power of the unit ball of M; the uniform continuity does not depend on the particular algebra $i.e.$ for every ϵ there is $a\ \delta$ independent of the choice of algebra;

(G2) For every ultrafilter $\mathcal{U}$ the function g can be canonically extended to the n-th power of the unit ball of the ultrapower $(M_{\leq 1})^{\mathcal{U}} = (M^{\mathcal{U}})_{\leq 1}$

The cardinality of the language andthe number of formulas are crude measures of the Loowenheim-Skolem cardinal for continuous logic. We define a topology on formulas relative to a given continuous theory in order to give a better measure.

Suppose T is a continuous theory in a language $\mathcal{L}$. Fix variables $\bar{x} = x_1 \dots x_n$ and domains $\bar{D} = D_1 \dots D_n$ consistent with the sorts of the $x's$. For formulas φ and ψ defined on $\bar{D}$, set

$$d_{\bar{D}}^{T}\big(\varphi(\bar{x}), \psi(\bar{x})\big) = \sup\left\{\sup_{\bar{x}\in(\bar{D}\mathcal{M})^n} |\varphi - \psi(\bar{x}) \colon \mathcal{M}| = T\right\}.$$

Now $d_{\bar{D}}^{T}$ is a pseudo-metric; let $\chi(T, \bar{D})$ be the density character of this pseudo-metric on the formulas in the variables $\bar{x}$ and define the density character of $\mathcal{L}$ with respect to $T, \chi(T)$, as $\sum_{\bar{D}} \chi(T, \bar{D})$.

We will say that $\mathcal{L}$ is separable if the density character of $\mathcal{L}$ is countable with respect to all $\mathcal{L}$ -theories. Note that the languages considered, in particular $\mathcal{L}_{T_r}$, and $\mathcal{L}_{C^*}$are separable.

Proposition(5.1.10)[302]: Assume $\mathcal{L}$ is a separable language. Then for every model $\mathcal{M}$ of $\mathcal{L}$ the set of all interpretations of formulas of $\mathcal{L}$ is separable in the uniform topology.

Proof: Since we are allowing all continuous real functions as propositional connectives the set of formulas is not countable. However, a straightforward argument using polynomials with rational coefficients and the Stone-Weierstrass theorem gives a proof.

The following is a version of the downward Lowenheim-Skolem theorem (cf. [303]). Some of its instances have been rediscovered and applied in the context of C^*-algebras (see, e.g., [319] or the discussion of SI properties in [100]). We use the notation $\chi(X)$, to represent the density character of a set X in some ambient topological space.

Theorem(5.1.11)[302]: Suppose that $\mathcal{M}$ is a metric structure and $X \subseteq M$. Then there is $\mathcal{N} \prec \mathcal{M}$ such that $X \subseteq M$ and $\chi(\mathcal{N}) \leq \chi(\mathrm{Th}(\mathcal{M})) + \chi(X)$.

Proof: Fix $\mathcal{F}$, a dense set of formulas, witnessing $\chi(\mathrm{Th}(\mathcal{M}))$. Define two increasing sequences $\langle X_n: n \in \mathbb{N} \rangle$and $\langle E_n: n \in \mathbb{N} \rangle$of subsets of M inductively so that:
(i) $X_0 = X$;
(ii) $E_n$is dense in X_n and $\chi(X_n) = |E_n|$ for all $n \in \mathbb{N}$;
(iii) $(\chi(X_n) \le \chi(\mathrm{Th}(M)) + \chi(X)$; and,

(iv) for every rational number r, formula $\varphi(x, \bar{y}) \in \mathcal{F}$, domain D in the sort of the variable x and $\bar{a} = a_1, \dots, a_k \in E_n$ where k is the length of $\bar{y}$, if $\mathcal{M} \vDash \inf_{x \in D} \varphi(x, \bar{a}) \le r$ then for every $n > 0$ there is $b \in X_{n+1} \cap D(\mathcal{M})$ such that $\mathcal{M} \vDash \varphi(b, \bar{a}) \le r + (1/n)$.
It is routine to check that $\overline{\bigcup_{n \in \mathbb{N}} X_n}$ is the universe of an elementary submodel $\mathcal{N} \prec \mathcal{M}$ having the correct density character.
Corollary (5.1.12)[302]: Assume $\mathcal{L}$ is separable. If $\mathcal{M}$ is a model of $\mathcal{L}$ and X is an infinite subset of its universe, then $\mathcal{M}$ has an elementary submodel whose density character is not greater than that of X and whose universe contains X.
Suppose that $\mathcal{M}$ is a model in a language $\mathcal{L}, A \subseteq M$ and $\bar{x}$ is a tuple of free variables thought of as the type variables.

We follow [303] and say that a condition over A is an expression of the form $\varphi(\bar{x}, \bar{a}) \le r$ where $\varphi \in \mathcal{L}, \bar{a} \in A$ and $r \in \mathbb{R}$. If $\mathcal{N} \succ \mathcal{M}$ and $\bar{b} \in N$if then $\bar{b}$satisfies $\varphi(\bar{x}, \bar{a}) \le r$ if $\mathcal{N}$ satisfies$\varphi(\bar{b}, \bar{a}) \le r$.

Fix a tuple of domains $\bar{D}$ consistent with $\bar{x}$, i.e., if x_{i} is of sort S then D_{i} is a domain in $S. A$ set of conditions over A is called a $\bar{D}$-type over A. A $\bar{D}$-type is consistent if for every finite $p_0 \subseteq p$ and $\epsilon > 0$ there is $\bar{b} \in \bar{D}(M)$ such that if $''\varphi(\bar{x}, \bar{a}) \le r'' \in p_0$ then M satisfies $\varphi(\bar{b}, \bar{a}) \le r + \epsilon$. We say that a $\bar{D}$-type p over A is realized in $\mathcal{N} \succ \mathcal{M}$ if there is $\bar{a} \in \bar{D}(N)$ such that $\bar{a}$ satisfies every condition in p. The following proposition links these two notions:
Proposition(5.1.13)[302]: The following are equivalent:
(i) p is consistent.
(ii) p is realized in some $\mathcal{N} \succ \mathcal{M}$.
(iii) p is realized in an ultrapower of $\mathcal{M}$.
Proof: (iii) implies (ii) and (ii) implies (i) are clear. To see that (i) implies (iii), let $F \subseteq p \times \mathbb{R}_+$ be a finite set, and let $\bar{b}_F \in \bar{D}(M)$ satisfy $\varphi(\bar{x}, \bar{a}) \le r + \epsilon$ for every $(\varphi(\bar{x}, \bar{a}) \le r, \epsilon) \in F$. Let $\mathcal{U}$ be a non-principal ultrafilter over $P_{fin}(p \times \mathbb{R}_+)$. Then p is realized by $(\bar{b}_F: F \in \mathcal{P}_{fin}(p \times \mathbb{R}_+)) / \mathcal{U}$ in $\mathcal{M}^{\mathcal{U}}$.

A maximal consistent $\bar{D}$-type is called complete. Let $S^{\bar{D}}(A)$ be the set of all complete$\bar{D}$-types over A. In fact, p is a complete$\bar{D}$-type over A iff p is the set of all conditions true for some $\bar{a} \in \bar{D}(\mathcal{N})$ where $\mathcal{N} \succ \mathcal{M}$.
Notation(5.1.14)[302]: Assume p is a complete type over A and $\phi(x, \bar{\mathrm{a}})$ is a formula with parameters $\bar{\mathrm{a}}$ in A. Since p is consistent and maximal, there is the unique real number $r = \sup\{s \in \mathbb{R}$: the condition $\phi(x, \bar{\mathrm{a}}) \le s$ is in $p\}$. In this situation we shall extend the notation by writing $\phi(x, \bar{\mathrm{a}}) = r$. We shall also use expressions such as $\left|\phi(x, \bar{a}) \ \_\phi(p, \bar{b})\right| > \varepsilon$.

We will also often omit the superscript $\bar{\mathrm{D}}$ when it either does not matter or is implicit. The set $S^{\bar{\mathrm{D}}}(A)$ carries two topologies: the logic topology and the metric topology.

Fix $\varphi, \bar{a} \in A$ and $r \in \mathbb{R}$. A basic closed set in the logic topology has the form

$$\{p \in S^{\bar{D}}(A): \varphi(\bar{x}, \bar{a}) \leq r \in p\}$$

The compactness theorem shows that this topology is compact and it is straightforward that it is Hausdorff.

We can also put a metric on $S^{\bar{D}}(A)$ as follows: for $p, q \in S^{\bar{D}}(A)$ define $(p, q) = \inf\{d(a, b)$: there is an $\mathcal{N} \succ \mathcal{M}, a$ realizes p and b realizes $q\}$:

The metric topology is in general finer than the logic topology due to the uniform continuity of formulas.

Example(5.1.15)[302]: Let M be a model corresponding to a tracial von Neumann algebra or a unital C^*-algebra.

(i)The relative commutant type of $\mathcal{M}$ is the type over M consisting of all conditions of the form

$$d([a, x], 0) = 0$$

(ii) Another type over $\mathcal{M}$ consists of all conditions of the form

$$d(a, x) \geq \varepsilon$$

for $a \in M$ and a fixed $\varepsilon > 0$.

While the relative commutant type is trivially realized by the center of $\mathcal{M}$, the type described in (ii) is never realized in $\mathcal{M}$. However, the second type is sometimes consistent. For instance, if $\mathcal{M}$ is an infinite dimensional C^*-algebras then (ii) is consistent. Hence not every consistent type over $\mathcal{M}$ is necessarily realized in $\mathcal{M}$.

A model $\mathcal{M}$ of language $\mathcal{L}$ is countably saturated if for every countable subset X of the universe of $\mathcal{M}$, every consistent type over X is realized in $\mathcal{M}$. More generally, if $\mathcal{K}$ is a cardinal then $\mathcal{M}$ is k-saturated if for every subset X of the universe of $\mathcal{M}$ of cardinality $< K$every consistent type over X is realized in $\mathcal{M}$. We say that $\mathcal{M}$ is saturated if it is $\mathcal{K}$-saturated where $\mathcal{K}$ is the density character of $\mathcal{M}$.

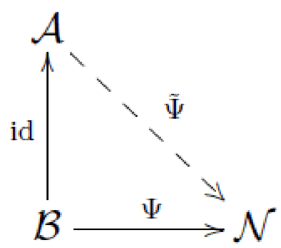
Thus countably saturated is the same as $\aleph_1$-saturated, where $\aleph_1$is the least uncountable cardinal. The following is a version of a classical theorem of Keisler for the logic of metric structures.

Proposition(5.1.16)[302]: If $\mathcal{M}_i$, for $i \in \mathbb{N}$, are models of the same language and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ then the ultraproduct $\prod_{\mathcal{U}} \mathcal{M}_i$ is countably saturated. If $\mathcal{M}$ is separable then the relative commutant of $\mathcal{M}$ in $\mathcal{M}^{\mathcal{U}}$ is countably saturated.

Proof: A straightforward diagonalization argument, cf. the proof of Proposition 4.8.

The following lemma is a key tool.

Lemma(5.1.17)[302]: Assume $\mathcal{N}$ is a countably saturated $\mathcal{L}$-structure, A and B are separable $\mathcal{L}$ structures,and B is an elementary submodel of A.Also assumeψ: $\mathcal{B} \to \mathcal{N}$is an elementary embedding. Then Ψ can be extended to an elementary embedding $\Phi: \mathcal{A} \to \mathcal{N}$.



Proof: Enumerate a countable dense subset of A as a_n, for $n \in \mathbb{N}$, and fix a countable dense $B_0 \subseteq B$. Let t_n be the type of $a_n$over $B_0 \cup \{a_j: j < n\}$. If t is a type over a subset X of A

then by $\Psi(t)$ we denote the type over the Ψ -image of X obtained from t by replacing each $a \in A$ by $\Psi(a)$. By countable saturation realize $\Psi(t_0)$ in $\mathcal{N}$ and denote the realization by $\Psi(a_0)$ in order to simplify the notation. The type $\Psi(t_1)$is realized in $\mathcal{N}$ by an element that we denote by (t_1) . Continuing in this manner, we find elements $\Psi(a_n)$in $\mathcal{N}$for $n \in \mathbb{N}$. Since the sequence a_{n}, for $n \in \mathbb{N}$, is dense in A, by elementarity the map $a_n \to \Psi(a_n)$ can be extended to an elementary embedding $\Phi: \mathcal{A} \longrightarrow \mathcal{N}$ as required.

The analogue of Lemma(5.1.17) holds when, instead of assuming $\mathcal{A}$ to be separable,$\mathcal{N}$ is assumed to be $\mathcal{K}$-saturated for some cardinal $\mathcal{K}$greater than the density characterof $\mathcal{A}$. Using a transfinite extension of Cantor's back-and-forth method, Proposition(5.1.12) and this analogue of Lemma(5.1.17) one proves the following.

Proposition(5.1.18)[302]: Assume $\mathcal{L}$ is a separable language. If $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent saturated models of $\mathcal{L}$ that have the same uncountable density character then they are isomorphic.

We refer to tracial von Neumann algebras (C^*algebras, unitary groups of a tracial von Neumann algebra or a C^*-algebra, respectively) as`algebras.'

Corollary(5.1.19)[302]: Assume the Continuum Hypothesis. If M is an algebra of density character $\leq c$ then all of its ultrapowers associated with nonprincipal ultrafilters are isomorphic. If M is separable, then all of its relative commutants in ultrapowers associated with nonprincipal ultrafilters are isomorphic.

Proof: The Continuum Hypothesis implies that such ultrapowers are saturated and by Proposition(5.1.8), Proposition(5.1.16) and Proposition(5.1.18) they are all isomorphic. If M is separable, then the isomorphism between the ultrapowers can be chosen to send the diagonal copy of M in one ultrapower to the diagonal copy of M in the other ultrapower and therefore the relative commutants are isomorphic.

It should be noted that, even in the case when the Continuum Hypothesis fails, countable saturation and a transfinite back-and-forth construction together show that ultrapowers of a fixed algebra are very similar to each other.

Corollary(5.1.20)[302]: Assume M is a separable algebra and $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters on $\mathbb{N}$. Then for all separable algebras N we have the following:

(i) N is a subalgebra of $M^{\mathcal{U}}$ if and only if it is a subalgebra of $M^{\mathcal{V}}$;

(ii) N is a subalgebra of $M' \cap M^{\mathcal{U}}$ if and only if N is a subalgebra of $M' \cap M^{\mathcal{V}}$.

Proof: These classes of algebras are axiomatizable, so instead of algebras we can work with the associated models. Supposing that $\mathcal{N} \subset \mathcal{M}^{\mathcal{U}}$, apply the downward Lowenheim-Skolem theorem (Proposition (5.1.13) to find an elementary submodel $\mathcal{P}$ of $\mathcal{M}^{\mathcal{U}}$ whose universe contains N and the diagonal copy of M. Now consider the elementary inclusion $\mathcal{M} \subseteq \mathcal{P}$, and use Lemma(5.1.17) to extend the map which identifies $\mathcal{M}$ with the diagonal subalgebra of $\mathcal{M}^{\mathcal{V}}$, the latter being countably saturated by Proposition(5.1.16). This extension carries P onto asubalgebra of $M^{\mathcal{V}}$ and restricts to an isomorphism from N onto its image. In case M and N commute, their images in $\mathrm{M}^{\mathcal{V}}$ do too.

We also record a refining of the fact that the relative commutants of a separable algebra are isomorphic assuming CH.

Corollary(5.1.21)[302]: Assume $M, \mathcal{V}$ and $\mathcal{U}$ are as in Corollary(5.1.20). Then the relative commutants $M' \cap M^{\mathcal{U}}$ and $M' \cap M^{\mathcal{V}}$ are elementarily equivalent.

Proof: By countable saturation of ultrapowers, a type p over the copy of M inside $M^{\mathcal{U}}$ is realized if and only the same type over the copy of M inside $M^{\mathcal{V}}$ is realized. By considering only types p that extend the relative commutant type the conclusion follows.

The conclusion of Corollary(5.1.21) fails when M is the C^*-algebra $\mathcal{B}(\ell^2)$. By [309] CH implies $\mathcal{B}(\ell^2)' \cap \mathcal{B}(\ell^2)^{\mathcal{U}}$ is trivial for one $\mathcal{U}$ and infinite-dimensional for another.This implies that the assumption of separability is necessary in Corollary (5.1.21).

This defines the two main model theoretic notions: stability and the order property. We show that each is equivalent to the negation of the other Theorem (5.1.26), and that the order property is equivalent to the existence of nonisomorphic ultrapowers when the continuum hypothesis fails Theorem (5.1.27). While the analogue of the former fact is well-known in the discrete case, we could not find a reference to the analogue of the latter fact in the discrete case. We have already seen that when the continuum hypothesis holds all ultrapowers are isomorphic Corollary (5.1.19).

Definition(5.1.22)[302]: We say a theory T is $\curlywedge$-stable if for any model M of T of density character $\curlywedge$, the space of complete types $S(M)$ has density character $\curlywedge$ in the metric topology on $S(M)$. We say T is stable if it is stable for some $\curlywedge$ and T is unstable if it is not stable.

For a theory T in a separable language one can show that T is stable if and only if it is c-stable (see the proof of Theorem (5.1.2)).

Our use of the term "stable" in agrees with model theoretic terminology in both continuous and discrete logic. Motivated by model theory, in 1981 Krivine and Maurey de_ned a related notion of stability for Banach spaces that is now more familiar to many analysts ([316]). It is characterized by the requirement

(*) $$\lim_{i\to\mathcal{U}}\lim_{i\to\mathcal{V}}\|x_i + y_j\| = \lim_{i\to\mathcal{U}}\lim_{i\to\mathcal{V}}\|x_i + y_j\|,$$

for any uniformly bounded sequences $\{x_\mathrm{i}\}$ and $\{y_j\}$ and any free ultrafilters $\mathcal{U};\mathcal{V}$ on $\mathbb{N}$. One can show ([314]) that a Banach space satisfies (*) if and only if no quantifier-free formula has the order property in that structure, so model theoretic stability of the theory of a Banach space X implies stability of X in the sense of Krivine-Maurey.

We proved in [120] that all infinite-dimensional C^*-algebras are unstable. The same cannot be said for infinite-dimensional Banach algebras: take a stable Banach space and put the zero product on it. However a stable Banach space can become unstable when it is turned into a Banach algebra. We exhibit this behavior in Proposition(5.1.29) below.

Definition(5.1.23)[302]: We say that a continuous theory T has the order property if there is a formula $\psi(\bar{x},\bar{y})$ with the lengths of $\bar{x}$ and $\bar{y}$ the same, and a sequence of domains $\bar{D}$ consistent with the sorts of $\bar{x}$ and $\bar{y}$, and. a model M of T and $\langle \bar{a}_i : i \in \mathbb{N}\rangle \subseteq \bar{D}(M)$ such that

$$\psi(a_i, a_j) = \text{if } i < j \text{and } \psi(a_i, a_j) = 1 \text{ if } i \geq j.$$

Note that these evaluations are taking place in M. Also note that by the uniform continuity$\varepsilon = 0$ such that $d(\bar{a}_i, \bar{a}_j) \geq \varepsilon$ for every $i \neq j$ where the metric here is interpreted as the supremum of the coordinatewise metrics.

Proposition(5.1.24)[302]: Th(A) has the order property if and only if there is ψ and $\bar{D}$ such that for all n and $\delta > 0$, there are $a_1, \ldots, a_n \in \bar{D}(A)$ such that

$$\psi(a_i, a_j) \leq \delta \text{ if } i < j \text{ and } \psi(a_i, a_j) \geq 1 - \delta \text{ if } i \geq j.$$

Definition(5.1.25)[302]: Suppose that M is a metric structure and $p(\bar{x}) \in S^{\bar{D}}(M)$ is a type. We say that p is finitely determined if for every formula $\varphi(\bar{x}, \bar{y})$, choice of domains $\bar{D}'$ consistent with the variables $\bar{y}$, and $m \in \mathbb{N}$, there is $k \in \mathbb{N}$ and a finite set $B \subseteq \bar{D}(M)$ such that for every $\bar{c}_1, \bar{c}_2 \in \bar{D}'(M)$.

$$\lim_{\bar{b}\in B}\left|\varphi\left(\bar{b}, \bar{c}_1\right) - \varphi\left(\bar{b}, \bar{c}_1\right)\right| \leq \frac{1}{k} \Rightarrow |\varphi(p, \bar{c}_1) - \varphi(p, \bar{c}_1)| \leq \frac{1}{m}.$$

Theorem(5.1.26)[302]: The following are equivalent for a continuous theory T:

(i) T is stable.

(ii) T does not have the order property.

(iii) If M is a model of T and $p \in S(M)$ then p is finitely determined.

Proof: (i) implies (ii) is standard: suppose that T has the order property via a formula$\theta$and choose any cardinal $\curlywedge$. Fix $\mu \leq \curlywedge$ least such that $2^{\mu} > \curlywedge$ (note that $2^{<\mu} \leq \curlywedge$. By compactness, using the order property, we can find $\langle \bar{a}_i : i \in 2^{<\mu}\rangle$such that $\theta\left(\bar{a}_i, \bar{a}_j\right) = 0$ if $i < j$ in the standard lexicographic order and 1 otherwise. Clearly, $\chi(S(A)) > \chi(A)$ where $A = \{\bar{a}_i : i \in 2^{<\mu}\}$ so T is not $\curlywedge$-stable for any $\curlywedge$.

To see that (iii) implies (i), fix a model M of T with density character $\curlywedge$ where $\curlywedge^{\chi(T)} = \curlywedge$ By assumption, every type over M is finitely determined and so there are at most $\curlywedge^{\chi(T)} = \curlywedge$ many types over M and so T is $\curlywedge$-stable.

Finally, to show that (ii) implies (iii), suppose that there is a type over a model of T which is not finitely determined. Fix $p(\bar{x})\ 2\ S^{\bar{D}}(M), \varphi(\bar{x}, \bar{y})$, domains $\bar{D}'$ consistent with the variables $\bar{y}$ and $m \in \mathbb{N}$ so that for all k and finite $B \subseteq \bar{D}(M)$, there are $n_1, n_1 \in \bar{D}'(M)$such that

$$\max_{b\in B}|\varphi(b, n_1) - \varphi(b, n_2)| \leq 1 / k$$

but

$$|\varphi(b, n_1) - \varphi(b, n_2)| > 1/m .$$

We now use this p to construct an ordered sequence. Define sequences $a_j, b_j c_j$ and sets B_j as follows: $B_0 = \emptyset$. If we have defined B_{j} , choose $b_j, c_j \subseteq \bar{D}'(M)$ such that $max_{b\in B_j}\left|\varphi\left(b, b_j\right) - \varphi(b, c_j)\right| \leq 1 / 2m$ but $\left|\varphi\left(p, b_j\right) - \varphi(p, c_j)\right| \leq 1 / m$.

Now choose $a_j \in \bar{D}(M)$ so that a_{j} realizes $\varphi\left(\bar{x}, b_j\right) - \varphi(p, b_j)$ and $\varphi(\bar{x}, c_i)$ for all $i \leq j$. Let $B_{j+1} = B_j\left\{a_j, b_j, c_j\right\}$. It follows that if $i \geq j$ then $\left|\varphi\left(a_i, b_j\right) - \varphi(a_i, c_j)\right| \leq 1 / m$. If $i \geq j$ then $\left|\varphi\left(a_i, b_j\right) - \varphi(a_i, c_j)\right| \leq 1 / 2m$ since $a_i \in B_j$. Consider the formula

$$\theta(x_1, y_1, z_1, x_2, y_2, z_2) - |\varphi(x_1, y_2) - \varphi(x_1, z_2)|.$$

Then θ orders $\langle a_i, b_i, c_i : i \in \mathbb{N}\rangle$

Theorem(5.1.27)[302]: Suppose that A is a separable metric structure in a separable language.

(i) If the theory of A is stable then for any two non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}, A^{\mathcal{U}} \cong A^{\mathcal{V}}$.

(ii) If the theory of A is unstable then the following are equivalent:

(a) A has fewer than $2^{2^{\aleph_0}}$nonisomorphic ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$.

(b) for any two non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}, A^{\mathcal{U}} \cong A^{\mathcal{V}}$;

(c) the Continuum Hypothesis holds.

It is worth mentioning that Theorem(5.1.27) is true in the first order context, as can be seen by considering a model of a first-order theory as a metric model with respect to the discrete metric. Although this is undoubtedly known to many, we were unable to find a direct reference. The proof of (i) will use tools from stability theory, see [305].

Proof: (i) Assume that the theory of A is stable. We will show that $A^{\mathcal{U}}$ is c-saturated and so it will follow that $A^{\mathcal{U}} \cong A^{\mathcal{V}}$ no matter what the size of the continuum is (see Proposition (5.1.18)).

So suppose that $B \subseteq A^{\mathcal{U}}, |B| < c,$, and q is a type over B. We may assume that B is an elementary submodel and that q is nonprincipal and complete. As the theory of A is stable, choose a countable elementary submodel $B_0 \subseteq B$ so that q does not fork over B_0. We shall show that in $A^{\mathcal{U}}$ one can always find a Morley sequence in $q|_{B_0}$ of size c.

Towards this end, fix a countable Morley sequence I in the type of $q|_{B_0}$ and let $\bar{q} = tp(I / B_0)$, a type in the variables x_{n} for $n \in \mathbb{N}$. Since B_0 is countable and the language is separable, there are countably many formulas $\psi_n(x_1, \dots, x_n, b_n)$ over B_0 such that $\psi_n(x_1, \dots, x_n, b_n) = 0 \in \bar{q}$ and $\{\psi_n(x_1, \dots, x_n, b_n) = 0 : n \in \mathbb{N}\}$ axiomatizes $\bar{q}$.

Now let $D_i = \{n \geq i : \inf_x \psi_i \left(x, b_i(n)\right) < 1 / i\}$; For a fixed n, consider $\{i : n \in D_i\}$. This set has a maximum element; call it i_{n}. Now fix $a_1^n, \dots, a_{i_n}^n \in A$ such that $\psi_{i_n} = \left(a_1^n, \dots, a_{i_n}^n, b_{i_n}\right) < 1 / i_n$. Now consider the set J of all $g : \mathbb{N} \to A$ such that $g(n) \in \left\{a_1^n, \dots, a_{i_n}^n\right\}$ for all n. Any $g \in J$ will satisfy $q|_{B_0}$in $A^{\mathcal{U}}$ since every element of I realized that type. If $g_0, \dots, g_k$ are in J and distinct modulo U then they are independent over B_0 since I was a Morley sequence. To finish then, we need to see that there are c-many distinct $g\,'s$ modulo U.

This follows from the fact that the in0 $'s$ are not bounded. To see this, for a fixed m, let $X = \{n \geq m : \inf_x \psi_m(x, b(n)) \leq 1 / m\}$. Pick any $n \in X$. We have that $n \in D_m$ So $i_n \geq m$ and we conclude that the $i_n\,'s$ are not bounded.

Since $|B| < c,$ there is a J_0 of cardinality less than c such that B is independent from J over J_0. Choosing any $a \in J \setminus J_0$ and using symmetry of non-forking remembering that J is a Morley sequence over B_0 , it follows that a is independent from B over B_0 is a model, $q|_{B_0}$ has a unique non-forking extension to B and it follows that a realizes q. This nishes the proof of (i).

(ii) If the Continuum Hypothesis holds then $A^{\mathcal{U}}$ is always saturated and so for any two ultrafilters $\mathcal{U}, \mathcal{V}, A^{\mathcal{U}} \cong A^{\mathcal{V}}$ even if we fix the embedded copy of A (Corollary (5.1.19)).

The implication (a) implies (c) follows from [310] and of course (b) implies (a). However, (b) implies (c) also can be proved by a minor modification of proof from [120] (which is in turn a modification of a proof from [308]), so we assume that the reader has a copy of the former handy and we sketch the differences. Assume the theory of A is unstable. Then by Theorem(5.1.26) it has the order property. The formula witnessing the order property satisfies [120] by Corollary(5.1.9). Therefore the analogues of [120] can be proved by quoting their proofs verbatim.

Hence if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ then $K(\mathcal{U}) = \leftthreetimes$ (defined in [120]) if and only if there is a $(\aleph_0, \leftthreetimes) - \psi -$gap in $A^{\mathcal{U}}$. By [308], if CH fails then there are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $k(\mathcal{U}) \neq k(\mathcal{V})$ and this concludes the proof.

We include two examples promised earlier and state three rather different problems.

Recall that UHF, or uniformly hyperfinite, algebras are C^*-algebras that are C^*-tensor products of (finite-dimensional) matrix algebras. They form a subcategory of C^*-algebras and the morphisms between them are *-homomorphisms.

Proposition(5.1.28)[302]: The category of UHF algebras is not axiomatizable.

Proof: By Proposition(5.1.6) it will suffice to show that this category is not closed under taking (C^*-algebraic) ultraproducts. We do this by repeating an argument from Ge-Hadwin ([311]) exploiting the fact that UHF algebras have unique traces that are automatically faithful.

Let A be the CAR algebra $\otimes_{n\in\mathbb{N}} M_2(\mathbb{C})$ with trace tr, let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and let $\{p_n\} \subset A$ be projections with $tr(p_n) = 2^{-n}$. The sequence (p_n) represents a nonzero projection in $A^{\mathcal{U}}$, but $tr^{\mathcal{U}}((p_n)) = 0$. Thus $tr^{\mathcal{U}}$ is a non-faithful tracial state, so that $A^{\mathcal{U}}$ is not UHF.

The same argument shows that simple C^*-algebras are not axiomatizable.

Since every UHF algebra has a unique trace one could also consider tracial ultraproducts, instead of norm-ultraproducts, of UHF algebras. However, such an ultraproduct is always a factor ([313]) and therefore not a UHF algebra (because projections in UHF algebras have rational traces).

The L^{P} Banach spaces $(1 \le p < \infty)$ are known to be stable ([303]), and they become stable Banach algebras when endowed with the zero product. Actually $\ell^p (1 \le p < \infty)$ with pointwise multiplication is also stable; this can be shown by methods similar to [120]. We now prove that the usual convolution product turns $\ell^1(\mathbb{Z})$ into an unstable Banach algebra, as was mentioned.

Proposition(5.1.29)[302]: The branch algebra $\ell^1 = \ell^1(\mathbb{Z}, +)$ (with convolution product) is unstable.

Proof: It suffices to show the order property for ℓ^1 algebra. This means we must give a formula $\varphi(x, y)$ of two variables (or n-tuples) on ℓ^1, a bounded sequence $\{x_i\} \subset \ell^1$ and $\varepsilon > 0$ such that $\varphi(x_i, x_j) \le \varepsilon$ when $i \le j$ and $\varphi(x_i, x_j) \ge 2\varepsilon$ when $i > j$.

Let $\{f_n\}_{n\in\mathbb{Z}}$ denote the standard basis for ℓ^1, so that multiplication is governed by the rule $f_m f_n = f_{m+n}$. Also let $\ell^1 \ni x \mapsto \hat{x} \in C(\mathbb{T})$ be the Gel'fand transform on ℓ^1so that $\hat{f}_n$ is the function $\left[e^{it} \mapsto e^{int}\right]$. The Gel'fand transform is always a contractive homomorphism; on ℓ^1it is injective but not isometric.

We take $\varphi(x, y) = \inf_{\|z\|\le 1} \|xz - y\|$, $x_i = \left(\frac{f_1+f_{-1}}{2}\right)^{2^i}$, and $\varepsilon = \frac{1}{8}$. are unit vectors, being convolution powers of a probability measure on $\mathbb{Z}$, and $\hat{x}_i = \left[e^{it} \mapsto (\cos t)^{2^i} \in C(\mathbb{T})\right]$.

For $i \le j$, we have $\varphi(x_i, x_j) = 0$ by taking $z = (\frac{f_1+f_{-1}}{2})^{2^j-2^i}$.

For $i > j$, let $t_0 \in (0,2\pi)$ be such that $(\cos t_0)^{2^i}$ For any $z \in (\ell^1)_{\le 1}$

$$\|x_i z - x_j\|_{\ell^1} \ge \|\hat{x}_i \hat{z} - \hat{x}_j\|_{C(\mathbb{T})} \ge \left|\left(\frac{1}{2}\right)^{2^{i-j}} z(e^{it_0}) - \frac{1}{2}\right| \ge \frac{1}{4},$$

where the middle inequality is justified by evaluation at t_0. We conclude that $\varphi(x_i, x_j) \ge \frac{1}{4}$as desired.

A well-known problem of Brown{Douglas{Fillmore ([306]) asks whether there is an automorphism of the Calkin algebra that sends the image of the unilateral shift to its adjoint.

The main result of [104] implies that if ZFC is consistent then there is a model of ZFC in which there is no such automorphism. A deep metamathematical result of Woodin, known as the $\sum_1^2$ absoluteness theorem, essentially (but not literally) implies that the Brown-Douglas-Fillmore question has a positive answer if and only if the Continuum Hypothesis implies a positive answer (see [322]). The type referred to in the following question is the type over the empty set.

We end with discussion of finite-dimensional matrix algebras and a result that partially complements [120], where it was proved that if the Continuum Hypothesis fails then the matrix algebras $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, have nonisomorphic tracial ultraproducts.

Proposition(5.1.30)[302]: Every increasing sequence $n(i)$, for $i \in \mathbb{N}$, of natural numbers has a further subsequence $m(i)$, for $i \in \mathbb{N}$ such that if the Continuum Hypothesis holds then all tracial ultraproducts of $M_{m(i)}(\mathbb{C})$, for $i \in \mathbb{N}$, are isomorphic.

Proof: The set of all $\mathcal{L}$-sentences is separable. Let $T_n = \mathrm{Th}(M_n(\mathbb{C}))$, the map associating the value $\psi^{M_n}(\mathbb{C})$of a sentence ψ in $M_n(\mathbb{C})$ to ψ. Since the set of sentences is separable we can pick a sequence $m(i)$ so that the theories $T_{m(i)}$ converge pointwise to some theory T_∞. Let $\mathcal{U}$ be a nonprincipal ultrafilter such that $\{m(i): i \in \mathbb{N}\} \in \mathcal{U}$. By Los's theorem, Proposition(5.1.8), $\mathrm{Th}(\prod_{\mathcal{U}} M_n(\mathbb{C})) = T_\infty$. By Proposition(5.1.16) and the Continuum Hypothesis all such ultrapowers are saturated and therefore Proposition(5.1.18) implies all such ultrapowers are isomorphic.

Section (5.2): C^*-Algebras Shuhei Masumoto

A topological space is said to have the countable chain condition (CCC for short) if every family of mutually disjoint nonempty open subsets is countable.

Any separable space clearly has CCC. Conversely, every metric space which has CCC is separable.

The relation between separability and direct products is simple. The direct product of a family of separable spaces are separable when its cardinality is less than or equal to 2^ω; but if the cardinality of the family is greater than 2^ω, then its direct product can be nonseparable. On this point, CCC behaves differently: it is irrelevant to the cardinality of the family. It is known that the direct product of a family of CCC spaces has CCC if for every finite subfamily, its direct product has CCC; however, the statement that the direct product of two CCC spaces has CCC cannot be proved or disproved in ZFC [106].

Now we shall restrict our interest to locally compact Hausdorff spaces. Let be a locally compact Hausdorff space and $C_0(X)$ be the C^*-algebra of the continuous functions on X which vanish at infinity. In view of the Gelfand-Naimark theorem, $C_0(X)$ contains all the information about the topological structure of X.

In particular, there is a canonical one to one correspondence between the open sets of X and the closed ideals of $C_0(X)$, and CCC can be reformulated as a condition on the ideal structure of $C_0(X)$, whence this condition can be generalized for noncommutative C^*-algebras. Moreover, since $C_0(X \times Y)$ is canonically isomorphicto $C_0(X) \otimes C_0(Y)$, the discussion on the relation between CCC and direct products yields information about the ideal structure of tensor products of C^*-algebras. In this way, we prove the following theorems:

The precise definition of CCC for C^*-algebras is introduced. Martin's Axiom, which is known to be independent from ZFC, is explained. Here it is also verified that the negation of the Suslin Hypothesis, which is another independent statement explained, implies the opposite conclusion of Theorem (5.2.19). We prove Theorems (5.2.16) and (5.2.19). Combining this fact with Theorem (5.2.16), we conclude that the statement that tensor products of CCC C*-algebras has CCC is independent from ZFC.

Definition(5.2.1)[323]: Two nonzero ideals in a C^*-algebra are said to be orthogonal if their intersection is the zero ideal. A C^*-algebra has the countable chain condition (CCC) if any family of nonzero mutually orthogonal ideals is countable.

Note that if $I, \mathcal{J}$ are ideals in a C^*-algebra, then $I \cap \mathcal{J}$ coincides with $\overline{I\mathcal{J}}$, whence they are orthogonal if and only if $I\mathcal{J} = 0$.

We shall begin with verifying that this definition is a generalization of CCC for topological spaces. Recall that a topological space has CCC if any family of nonempty mutually disjoint open subsets is countable.

Proposition(5.2.2)[323]: Let X be a locally compact Hausdorff space. Then $C_0(X)$ has CCC as a C^*-algebra if and only if X has CCC as a topological space.

Proof: Suppose first that X has CCC and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of nonzero mutually orthogonal ideals in $C_0(X)$. We can take an element $f_\lambda \in I_\lambda$ of norm 1 for each λ. Set $U_\lambda = \{x \in X \mid |f_\lambda(x)| > 1/2\}$. Then $\{U_\lambda\}_{\lambda\in\Lambda}$ is a family of nonempty mutually disjoint open subsets of X, whence $\#\Lambda \leq \omega$. Thus, $C_0(X)$ has CCC by definition.

If X admits an uncountable family $\{U_\lambda\}_{\lambda\in\Lambda}$ of nonempty mutually disjoint open sets, then $\{C_0(U_\lambda)\}_{\lambda\in\Lambda}$ is an uncountable family of nonzero mutually orthogonal ideals of $C_0(X)$. Therefore, $C_0(X)$.does not have CCC.

The following easy proposition characterizes CCC. Note that a von Neumann algebra is said to be σ-finite if it admits no uncountable family of mutually orthogonal projections.

Proposition(5.2.3)[323]:

(i) Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ has CCC if and only if there exists no family $\{a_\lambda\}_{\lambda\in\Lambda}$ of nonzero elements such that $a_\lambda \mathcal{A} a_\mu = 0$ for $\lambda \neq \mu$.

(ii) A von Neumann algebra has CCC if and only if its center is σ-finite.

Proof: (i) Suppose that there is an uncountable family $\{a_\lambda\}_{\lambda\in\Lambda}$ of nonzero elements such that $a_\lambda \mathcal{A} a_\mu = 0$ for $\lambda \neq \mu$. For each $\lambda \in \Lambda$, let $\overline{\mathcal{A}a_\lambda\mathcal{A}}$ be the ideal generated by a_λ. Then $\{I_\lambda\}_{\lambda\in\Lambda}$ is an uncountable family of nonzero mutually orthogonal ideals, so $\mathcal{A}$ does not have CCC.

Conversely, assume that $\mathcal{A}$ does not have CCC and let $\{I_{\lambda\in\Lambda}\}$ be an uncountable family of nonzero mutually orthogonal ideals. Taking nonzero $a_\lambda \in I_\lambda$ for each λ, we obtain $a_\lambda \mathcal{A} a_\mu = 0$ for $\lambda \neq \mu$ because $I_\lambda I_\mu = 0$.

(ii) Let I_1, I_2 be ideals of a von Neumann algebra $\mathcal{M}$. Then it can be easily verified that $I_1 I_2 = 0$ if and only if $\bar{I}_1^{\sigma w}\bar{I}_2^{\sigma w} = 0$, where $\bar{I}_i^{\sigma w}$ denotes the σ-weak closure of I_i. Now $\bar{I}_i^{\sigma w}$ is of the form $\mathcal{M}_{\mathcal{Z}_i}$ for a central projection $\mathcal{Z}_i$, and the two ideals are orthogonal if and only if these projections are orthogonal.

Therefore, $\mathcal{M}$ has CCC if and only if there is no uncountable family of nonzero mutually orthogonal projections, that is, σ-finite.

Proposition(5.2.4)[323]: A separable C^*-algebra has CCC.

Proof. Suppose that $\mathcal{A}$ does not have CCC, and $\{I_\lambda\}_{\lambda\in\Lambda}$ be an uncountable family of nonzero mutually orthogonal ideals. If $h_\lambda \in I_\lambda$ is a positive element of norm 1, then it follows by functional calculus that $\|h_\lambda - h_\mu\| = 1$. If we denote by U_λ the open ball of radius $1/2$

centered at $h_{\lambda,}$ then $\{U_\lambda\}_{\lambda\in\Lambda}$ is an uncountable family of mutually disjoint open subsets. Hence, $\mathcal{A}$ is not separable.

An ideal of a CCC C^*-algebra clearly has CCC. Also, it can be easily verified
that an extension of a CCC C^*-algebra by a CCC C^*-algebra has CCC. On the other hand, a quotient of a CCC C^*-algebra does not necessarily have CCC. Indeed, let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$. It has CCC because it is separable.
However, the StoneČech remainder $\beta\mathbb{N} \setminus \mathbb{N}$ does not have CCC because there exists an almost disjoint family of 2^ω subsets of ω[106]. Therefore, $C(\beta\mathbb{N} \setminus \mathbb{N})$ does not have CCC, although it is the quotient of the CCC C^*-algebra $C(\beta\mathbb{N}) \simeq \ell^\infty$by $C_0(\mathbb{N}) \simeq c_0$.

Since $C(\beta\mathbb{N} \setminus \mathbb{N})$can be obtained as the inductive limit of $\ell^\infty \overset{\varphi}{\to} \ell^\infty \overset{\varphi}{\to} \cdots$,where $\varphi: \ell^\infty \to \ell^\infty$ is defined by $\varphi(f)(n) = f(n+1)$, it also follows that inductive limits of CCC C^*-algebras do not necessarily have CCC. On this direction, what we can prove is the following:

To prove this proposition, we use the lemma below. A proof can be found in [326].

Lemma(5.2.5)[323]: Let $\mathcal{A}$ be a C^*-algebra and $\{\mathcal{A}_a\}$ be a directed set of subalgebras with its union dense in $\mathcal{A}$. If I is an ideal of $\mathcal{A}$, then it is obtained as the closure of the union of $\{I \cap \mathcal{A}_\alpha\}$.

Proposition(5.2.6)[323]: Let $\mathcal{A}$be a C^*-algebra and κ be an infinite cardinal number with its cofinality not equal to ω_1. If there is an increasing sequence $\{\mathcal{A}_a\}_{a<k}$ of CCC C^*-subalgebras such that $\overline{\bigcup_{a<k} \mathcal{A}_k} = \mathcal{A}$, then $\mathcal{A}$ has CCC.

Proof. Assume that there is an uncountable family $\{I_\lambda\}_{\lambda<\omega_1}$
Of nonzero mutually orthogonal ideals of $\mathcal{A}$. For each λ, set

$$\beta_\lambda = \min\{\alpha < \kappa \mid I_\lambda \cap \mathcal{A}_\alpha \neq 0\},$$

which exists by Lemma (5.2.5), and write $\beta = \sup_\lambda \beta_\lambda$. if $\beta < \kappa$ holds, then $\{\mathcal{A}_\beta \cap I_\lambda\}_\lambda$ is an uncountable family of nonzero mutually orthogonal ideals, which contradicts to the fact that $\mathcal{A}_\beta$ has CCC. On the other hand, if $\beta < \kappa$, then the cofinality of κ is ω, whence there is an unbounded increasing sequence $\{\gamma_n\}_{n<\omega}$ in κ. Now the set S_n of $\lambda < \omega_1$ with $\beta_\lambda < \gamma_n$ is at most countable for each n, whence $\omega_1 = \#(\bigcup_n S_n) \leq \omega$, a contradiction.

Proposition(5.2.7)[323]: Let $\mathcal{M}$ and $\mathcal{N}$ be CCC von Neumann algebras. Then the tensor product $\mathcal{M}\bar{\otimes}\mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ as a von Neumann algebra also has CCC.

Proof: We shall denote by $\mathcal{Z}(\mathcal{M}), \mathcal{Z}(\mathcal{N})$ and $\mathcal{Z}(\mathcal{M}\bar{\otimes}\mathcal{N})$ the centers of $\mathcal{M}, \mathcal{N}$ and $\mathcal{M}\bar{\otimes}\mathcal{N}$ respectively. Recall that $\mathcal{Z}(\mathcal{M}\bar{\otimes}\mathcal{N})$ coincides with $\mathcal{Z}(\mathcal{M})\bar{\otimes}\mathcal{Z}(\mathcal{N})$ [329]. Hence it suffices to show that the tensor product of two abelian σ-finite von Neumann algebras is also σ-finite. To see this, note that every abelian von Neumann algebra is of the form $L^\infty(\mu)$ for some Radon measure μ [329], and it is σ -finite if and only if μ is σ -finite. Since $L^\infty(\mu)\bar{\otimes}L^\infty(\nu)$,being canonically isomorphic to$L^\infty(\mu\otimes\nu)$, is σ-finite if $L^\infty(\mu)$ and $L^\infty(\nu)$ are both σ-finite, the conclusion follows.

A compact Hausdorff space is a stonean space if the closure of every open set is open. Suppose that X is a stonean space and μ is a Borel measure on it. If for any increasing family $\{f_i\} \in C_\mathbb{R}(X)$ with $\sup f_i = f \in C_\mathbb{R}(X)$ the equality $\mu(f) = \sup \mu(f_i)$ holds, then μ is said to be normal. A stonean space is called a hyperstonean space if for any nonzero positive $f \in C_\mathbb{R}(X)$ there exists a normal Borel measure μ such that $\mu(f) > 0$. It is known that if X is hyperstonean, then $C(X)$ is a von Neumann algebra, and every abelian von Neumann algebra is of this form [329]. Combining this fact with the preceding proposition, we obtain the following result.

Corollary(5.2.8)[323]: The direct product of two hyperstonean CCC spaces has CCC.
Proof: Let X, Y be hyperstonean CCC spaces. It follows from Proposition(5.2.7) that the von Neumann tensor product $C(X)\bar{\otimes}C(Y)$ has CCC, and $C(X)\otimes C(Y)$, which is isomorphic to $C(X \times Y)$, is a C*-subalgebra of $C(X)\bar{\otimes}C(Y)$. By Proposition(5.2.3), it can be easily verified that any C^*-subalgebra of commutative $CCC\ C^*$-algebra hasCCC, whence $X \times Y$ has CCC.

We introduce two statements which are known to be independent from ZFC, see [106] or [327].

The first statement is Martin's axiom. We shall introduce some definitions related to partially ordered sets in order to express this axiom in a simple form.

Definition(5.2.9)[323]: Let P be a nonempty partially ordered set. Two elements $p, q \in P$ are incompatible if there is no $r \in P$ with $r \leq p$ and $r \leq q$. If there is no uncountable family of mutually incompatible elements in P, then P is said to have the countable chain condition (CCC).

As is easily verified, a C^*-algebra has CCC if and only if its nonzero ideals form a CCC partially ordered set, where the order is defined by inclusion. Similarly, a nonempty topological space has CCC if and only if the set of nonempty open subsets has CCC as a partially ordered set.

Definition(5.2.10)[323]: Let P be a partially ordered set.
(i) A subset $D \subset P$ is dense if for any $p \in P$ there is $q \in D$ with $q \leq p$.
(ii) A nonempty subset $F \subset P$ is called a filter on P if it satisfies the following:
(a) if p, q are in F, then there exists $r \in F$ with $r \leq p$ and $r \leq q$;
(b) if $p \in F$ and $q \geq p$, then $q \in F$.

Suppose that P is a nonempty partially ordered set and fix the topology generated by subsets of the form $\{q \in P\ |q \leq p\}$ for $p \in P$. Then P has CCC if and only if P has CCC as a topological space, and $D \subset P$ is dense if and only if it is dense as a topological subspace.

Now we shall see the exact statement of Martin's axiom MA. Let κ be a cardinal number.
$MA(\kappa)$: If P is a nonempty CCC partially ordered set and $\{D_\alpha\}_{\alpha \in k}$ is a family of dense subsets in P, then there exists a filter F on P such that $F \cap D_\alpha$ }Is not empty for all α.
MA: $MA(\kappa)$ holds for any κ with $\omega \leq \kappa < 2^\omega$.

It is known that $MA(\omega)$ holds (the Rasiowa-Sikorski lemma) and $MA(2^\omega)$ does not hold in ZFC, whence the Continuum Hypothesis CH trivially implies MA. On the other hand, MA is indeed consistent with $ZFC + \neg CH$. In particular,$ZFC + MA(\omega_1)$ is consistent.

The other statement we use is Suslin's Hypothesis SH. This hypothesis is related to characterization of the real line as an ordered set. Note that a totally ordered set with the following properties is order-isomorphic to the real line:
(i) unbounded; there does not exist minimum nor maximum element.
(ii) dense; there is an element between any two elements.
(iii) complete; every nonempty bounded subset has a supremum and an inffimum.
(iv) separable; there is a countable subset which is dense with respect to the usual order topology.

Definition(5.2.11)[323]: Let S be a totally ordered set which is unbounded, dense and complete. Then S is called a Suslin line if it is nonseparable but CCC as a topological space, where its topology is the usual order topology generated by open intervals.
SH: There does not exist a Suslin line.

In other words, SH claims that separability in the characterization of the real line above can be replaced by CCC. It is known that the diamond principle, which is a consequence of the axiom of constructibility $V = L$, implies$\neg SH$[328]. On the other hand, $MA(\omega_1)$ implies SH, whence SH is independent from ZFC.

Proposition(5.2.12)[323]:A Suslin line is a locally compact space.

Proof: It suffices to show that every bounded closed interval is compact. This can be verified by seeing that a proof for the Heine-Borel theorem can be applied to Suslin lines.

Given an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of a bounded closed interval $[a,b]$, we shall prove that $[a,b]$ can be covered by finitely many U_λ's. Note that we may assume each U_λ is an open interval.

Let X be the set of all $x \in [a,b]$ such that $[a,x]$ can be covered by finitely many U_λ's. Then X is not empty because a is in X , and so sup X exists by completeness.

It suffices to show that sup X belongs to X and coincides with b. For this, take $\lambda_0 \in \Lambda$ such that sup X is in U_{λ_0} . Then $X \cap U_{\lambda_0}$ contains some element, say c. Now $[a,c]$ can be covered by finitely many U_λ's, and $[c, \sup X]$ is included in U_{λ_0} , so sup X is in X. Also, for any $x \in U$, the interval $[a,x]$ can be covered by finitely many U_λ's, whence sup X must coincide with b.

The following proposition is from [106]. For the sake of completeness, we include the proof.

Proposition(5.2.13)[323]: If S is a Suslin line, then $S \times S$ does not have CCC.

Proof: By transfinite induction, we shall take a_a , $b_a, c_a \in S$ for $a < \omega_1$ so that

(i) $a_\alpha < b_\alpha < c_\alpha$;

(ii) $b_\beta \notin (c_\alpha, c_\alpha)$ for $\beta < \alpha$.

This can be carried over because for each $\alpha < \omega_1$, the set $\{b_\beta | \beta < \alpha\}$, being countable,is not dense in S. Setting $U_a := (a_a, b_a) \times (b_a, c_a)$, we obtain an uncountable family $\{U_a\}_{a\in\Lambda}$ of nonempty mutually disjoint open sets in $S \times S$.

Corollary(5.2.14)[323]: $\neg SH$ implies the existence of a unital commutative CCC C* -algebra $\mathcal{A}$ such that $\mathcal{A}\otimes\mathcal{A}$ does not have CCC.

Proof: Let S be a Suslin line and consider the one point compactification S^+ of S . Since S contains S as a dense subspace, it is a CCC space. On the other hand, $S^+ \times S^+$ does not have CCC because it contains $S \times S$. Now $\mathcal{A} = C(S^+)$ is a unital commutative CCC C^*-algebra, but $\mathcal{A}\otimes\mathcal{A}$, being isomorphic to $C(S^+ \times S^+)$, does not have CCC.

Here we shall prove Theorems (5.2.16) and (5.2.19). For the first theorem, we need the following combinatorial lemma, which is known as the Δ-system lemma. A proof can be found in any standard textbook on set theory in which the method of forcing is dealt with.

Lemma(5.2.15)[323]: (Δ-system lemma). Every uncountable family of finite sets includes an uncountable subfamily whose pairwise intersection is constant.

Theorem (5.2.16)[323]: The minimal tensor product of a family of unital CCC C*-algebras has CCC if for every finite subfamily, its minimal tensor product has CCC.

Proof. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of unital C^*-algebras such that for every finite $J \subset I$, the minimal tensor product $\otimes_{i\in J}\mathcal{A}_i$ has CCC. We shall prove that $\mathcal{A} := \otimes_{i\in J}\mathcal{A}_i$ also has CCC.

Suppose that, contrary to our claim, there exists an uncountable family $\{I_\lambda\}_{\lambda\in\Lambda}$ of nonzero mutually orthogonal ideals in $\mathcal{A}$. By Proposition(5.2.5), we can find a finite subset $J_\lambda \subset I$

for each $\lambda \in \Lambda$ such that $I_\lambda \cap \otimes_{i\in J}\mathcal{A}_i \neq 0$. By the Δ-system lemma, we may assume that there exists a set R such that $I_\lambda \cap I_\mu = R$ for any $\lambda \neq \mu$. Since the tensor products are minimal, $I_\lambda \cap \odot_{i\in J_\lambda}\mathcal{A}_i$ is not zero for each λ, where $\odot_{i\in J_\lambda}\mathcal{A}_i$ is the algebraic tensor products of $\mathcal{A}_i$'s. Take nonzero $f_\lambda \in I_\lambda \cap \odot_{i\in J_\lambda}\mathcal{A}_i$ for each λ. If R is empty, then $f_\lambda f_\mu \neq 0$ for $\lambda \neq \mu$, which contradicts with the assumption that I_λ and I_μ are orthogonal to each other. Therefore, f_λ is of the form $\sum_k g_\lambda^k \otimes h_\lambda^k$, where g_λ^k is in $\otimes_{i\in R}\mathcal{A}_i$ and $\{h_\lambda^k\}_k$ is a linearly independent set in $\otimes_{i\in J_\lambda/R}\mathcal{A}_i$. If $\lambda \neq \mu$, then the equality $I_\lambda I_\mu = 0$ implies that $g_\lambda^k a g_\mu^l = 0$ for all k, l and $a \in \otimes_{i\in R}\mathcal{A}_i$. Since for each λ there exists k with $g_\lambda^k \neq 0$,, it follows that $\otimes_{i\in R}\mathcal{A}_i$ does not have CCC by Proposition(5.2.3), which is a contradiction. Therefore, $\otimes_{i\in I}\mathcal{A}_i$ has CCC.

Corollary(5.2.17)[323]: Every minimal tensor product of unital separable C^*-algebras has CCC.

We use the following lemma.

Lemma (5.2.18)[323]: Suppose that $\mathcal{A}$ is a CCC C^*algebra and $\{I_\alpha\}_{\alpha<\omega}$ be a family of its ideals. Then $MA(\omega_1)$ implies that there exists an uncountable subfamily of the ideals which has the finite intersection property.

Proof: Set $\mathcal{J}_\alpha := \sum_{\gamma<\alpha} I_\gamma$. Then $\mathcal{J}_\alpha$ is a transfinite decreasing sequence of ideals of $\mathcal{A}$. We shall first show that there exists α_0 such that $\mathcal{J}_\beta$ is an essential ideal of $\mathcal{J}_{\alpha_0}$ for all $\beta > \alpha_0$. Suppose the contrary. Then we can find an transfinite increasing sequence $\{\beta_\delta\}_{\delta<\omega_1} \subset \omega_1$ such that the inclusion $\mathcal{J}_{\beta_{\delta+1}} \subset \mathcal{J}_{\beta_\delta}$ is not essential. In other words, there exists a nonzero ideal $\mathcal{K}_{\beta_\delta}$ of $\mathcal{J}_{\beta_\delta}$ such that $\mathcal{K}_{\beta_\delta} \cap \mathcal{J}_{\beta_{\delta+1}} = 0$.Now $\{\mathcal{K}_{\beta_\delta}\}_{\delta<\omega_1}$ is an uncountable family of mutually orthogonal ideals in $\mathcal{A}$, which is a contradiction.

Next, let P be the set of nonzero ideals in $\mathcal{J}_{\alpha_0}$. Then P has CCC as a partially ordered set, because an ideal of a CCC C^*-algebra has CCC. For each $\beta > \alpha_0$, we Set

$$D_\beta = \{p \in P \mid p \subset I_\gamma \text{ for some } \gamma \geq \beta\}$$

and claim that this is dense in P. To prove this, take an arbitrary $q \in P$. Then $q' := q \cap \mathcal{J}_\beta$ is not zero by the definition of α_0. Here, $\mathcal{J}_\beta$ is approximated by $\sum_{\gamma\in S} I_\gamma$, where $S \subset]\beta, \omega_1[$ is finite. By [297], $\sum_{\gamma\in S} I_\gamma$ is norm closed for each S, whence we can use Lemma (5.2.5) to conclude that q' is the inductive limit of $\{q \cap \sum_{\gamma\in S} I_\gamma\}_S$, and so there exists $\gamma > \beta$ with $q \cap I_\gamma \neq 0$. Since $q \cap I_\gamma$ is clearly in D_β, it follows that D_β is dense, as desired.

Now let F be a filter on P such that $F \cap D_\beta$ is not empty for all β, whose existence is guaranteed by $MA(\omega_1)$. Then $\{I_\alpha \mid \exists p \in F, p \subset I_\alpha\}$ has the finite intersection property, and this is uncountable because the condition $F \cap D_\beta \neq \emptyset$ for each β implies that the set of all a such that $I_\alpha \supset p$ for some $p \in F$ is unbounded in ω_1.This completes the proof .

Theorem (5.2.19)[323] Martin's Axiom, $MA(\omega_1)$, implies that any minimal tensor product of unital CCC C*-algebras has CCC.

Proof. By Theorem (5.2.16), it suffices to show that if $\mathcal{A}$ and $\mathcal{B}$ have CCC, then $\mathcal{A} \otimes \mathcal{B}$ has CCC. Assume that, on the contrary, there exists a family $\{I_\alpha\}_{\alpha<\omega_1}$ of nonzero mutually orthogonal ideals in $\mathcal{A} \otimes \mathcal{B}$. Then there exist nonzero ideals $\mathcal{J}_\alpha \subset \mathcal{A}$ and $K_\alpha \subset \mathcal{B}$ with $\mathcal{J}_\alpha \odot K_\alpha \subset I_\alpha$, by [324]. Here, by the preceding lemma, we may assume that $\{\mathcal{J}_\alpha\}_\alpha$ and $\{\mathcal{K}_\alpha\}_\alpha$

satisfy the finite intersection property. Then, $I_\alpha \cap I_\beta$ contains $(\mathcal{J}_\alpha \cap \mathcal{J}_\beta) \otimes (\mathcal{K}_\alpha \cap \mathcal{K}_\beta) \neq 0$, which is a contradiction. Therefore, $\mathcal{A} \otimes \mathcal{B}$ has CCC, as expected.

Let $\mathcal{A}$ be a C^*-algebra. By Prim$(\mathcal{A})$, we shall denote the primitive spectrum of $\mathcal{A}$. (For the definition and elementary properties of primitive spectra, see [83].) It can be easily verified that $\mathcal{A}$ has CCC if and only if Prim$(\mathcal{A})$ has CCC as a topological space, and Lemma (5.2.18) is obtained as a corollary of [106]. Here, we may replace Prim$(\mathcal{A})$ by the prime spectrum prime$(\mathcal{A})$, because the topologies of these spaces are isomorphic as partially ordered sets.

In [330], it is proved that Prim$(\mathcal{A}\otimes\mathcal{B})$ is homeomorphic to Prim$(\mathcal{A}) \times$ Prim$(\mathcal{B})$ provided that either $\mathcal{A}$ or $\mathcal{B}$ is type I. Also, in [324], one can find various conditions for prime$(\mathcal{A}\otimes\mathcal{B}$ to be homeomorphic to prime $(\mathcal{A}) \times$ Prim$(\mathcal{B})$. In these cases, Theorem (5.2.19) follows from the corresponding fact for topological spaces [106].

One problem is whether Theorem (5.2.16) and Theorem (5.2.19) can be generalized to non-minimal tensor products. Since any tensor product has the minimal tensor product as its quotient, it depends on whether the kernel of the quotient map, which is difficult to be investigated, has CCC.

Another problem lies in the definition of CCC. We have defined CCC in terms of ideals, whence this condition is trivial for simple C^*-algebras.

In order to avoid this phenomenon, we can use hereditary C^*-algebras in place of ideals: we shall say two hereditary C^*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ are orthogonal to each other if $\overline{\mathcal{A}\mathcal{B}} = 0$; a C^*-algebra has strong CCC if there is no uncountable family of nonzero mutually orthogonal hereditary C^*-subalgebras. Then we can prove the following.

(a) Strong CCC implies CCC.

(b) C^*-subalgebras of a strong CCC C^*-algebra have strong CCC.

(c) An extension of a strong CCC C^*-algebra by a strong CCC C^*-algebra has strong CCC.

(d) A von Neumann algebra has strong CCC if and only if it is σ-finite, so tensor products of two strong CCC von Neumann algebras have strong CCC.

Corollary(5.2.20)[370]:

(i) Let $\mathcal{A}^2$ be a C^*-algebra. Then $\mathcal{A}^2$ has CCC if and only if there exists no family $\{a_\lambda\}_{\lambda\in\Lambda}$ of nonzero elements such that $\sum_r a_\lambda^r \mathcal{A}^2 a_{\lambda+\epsilon}^r = 0$ for $\epsilon \neq 0$.

(ii) A von Neumann algebra has CCC if and only if its center is σ-finite.

Proof: (i) Suppose that there is an uncountable family $\{a_\lambda^r\}_{\lambda\in\Lambda}$ of nonzero elements such that $\sum_r a_\lambda^r \mathcal{A}^2 a_{\lambda+\epsilon}^r = 0$ for $\epsilon \neq 0$. For each $\lambda \in \Lambda$, let $\overline{\mathcal{A}^2 a_\lambda^r \mathcal{A}^2}$ be the ideal generated by a_λ^r. Then $\{I_\lambda\}_{\lambda\in\Lambda}$ is an uncountable family of nonzero mutually orthogonal ideals, so $\mathcal{A}^2$ does not have CCC.

Conversely, assume that $\mathcal{A}^2$ does not have CCC and let $\{I_{\lambda\in\Lambda}\}$ be an uncountable family of nonzero mutually orthogonal ideals. Taking nonzero $a_\lambda^r \in I_\lambda$ for each λ, we obtain $\sum_r a_\lambda^r \mathcal{A}^2 a_{\lambda+\epsilon}^r = 0$ for $\epsilon \neq 0$ because $I_\lambda I_{\lambda+\epsilon} = 0$.

(ii) Let I_1, I_2 be ideals of a von Neumann algebra $\mathcal{M}$. Then it can be easily verified that $I_1 I_2 = 0$ if and only if $\bar{I}_1^{\sigma w} \bar{I}_2^{\sigma w} = 0$, where $\bar{I}_i^{\sigma w}$ denotes the σ-weak closure of I_i. Now $\bar{I}_i^{\sigma w}$ is of the form $\mathcal{M}_{\mathcal{Z}_i}$ for a central projection $\mathcal{Z}_i$, and the two ideals are orthogonal if and only if these projections are orthogonal.

Section (5.3): Elementary Equivalence of C^*-Algebras

We examine extent to which several classes of operator algebras are saturated in the sense of model theory. In fact, few operator algebras are saturated in the full model-theoretic sense, but in this setting there are useful weakenings of saturation that are enjoyed by a variety of algebras. The main results show that certain classes of C^*-algebras do have some degree of saturation, and as a consequence, have a variety of properties previously considered in the operator algebra. For all the definitions involving continuous model theory for metric structures (and in particular of C^*-algebras), see [335] or [119]. Different degrees of saturation and relevant concepts will be defined.

Among the weakest possible kinds of saturation an operator algebra may have, which nevertheless has interesting consequences, is being countably degree-1 saturated. This property was introduced by Farah and Hart in [114], where it was shown to imply a number of important consequences (see Theorem (5.3.11) below). It was also shown in [114] that countable degree-1 saturation is enjoyed by a number of familiar algebras, such as coronas of σ-unital C^*-algebras and all non-trivial ultraproducts and ultrapowers of C^*-algebras. Further examples were found by Voiculescu [37]. Countable degree-1 saturation can thus serve to unify proofs about these algebras. We extend the results of Farah and Hart by showing that a class of algebras which is broader than the class of σ-unital ones have countably degree-1 saturated coronas. The following theorem is Theorem(5.3.27) below; for the definitions of σ-unital C^*-algebras and essential ideals, see Definition(5.3.20).

Theorem (5.3.1). Let M be a unital C^*-algebr $a,$ and let $A \subseteq M$ be an essential ideal. Suppose that there is an increasing sequence of positive elements in A whose supremum is $1_m,$ and suppose that any increasing uniformly bounded sequence converges in M. Then M/A is countably degree-*1* saturated.

One interesting class of examples of a non-σ-unital algebra to which our result applies is the following. Let N be a II_1 factor, H a separable Hilbert space and $\mathcal{K}$ be the unique two-sided closed ideal of the von Neumann tensor product $N\overline{\otimes}\mathfrak{B}(\mathrm{H})$(see [337] and [338]). Then $(N\overline{\otimes}\mathfrak{B}(\mathrm{H}))/\mathcal{K}$ is countably degree-1 saturated.

We consider generalized Calkin algebras of uncountable weight, as well as $\mathfrak{B}(H)$ where H has uncountable density. Considering their complete theories as metric structures.

Theorem (5.3.2). Let $\alpha \neq \beta$ be ordinals, H_α the Hilbert space of density $\varkappa_\alpha$. Let $\mathfrak{B} = \mathfrak{B}(H_\alpha)$ and $C_\alpha = \mathfrak{B}_\alpha/\mathcal{K}$ the Calkin algebra of density. $\varkappa_\alpha$ Then the projections of the algebras $C_\alpha$and $C_\beta$as posets with respect to the Murray-von Neumann order are elementary equivalent if and only if$\alpha = \beta$ mod $\omega^\omega,$ where ω^ω is computed by ordinal exponentiation, as they are the infinite projections of $\mathfrak{B}_\alpha$ and $\mathfrak{B}_\beta$. Consequently, if $\alpha \not\equiv \beta$ then $\mathfrak{B}_\alpha \not\equiv \mathfrak{B}_\beta$ and $C_\alpha \not\equiv \mathfrak{B}_\beta$.

Elementary equivalence of C^*-algebras A and B can be understood, via the Keisler-Shelah theorem for metric structures, as saying that A and B have isomorphic ultrapowers.

For our second group of results we consider (unital) abelian C^*-algebras, which are all of the form $C(X)$ for some compact Hausdorff space X. We focus in particular on the real rank zero case, which corresponds to X being 0-dimensional. We first establish a correspondence between the Boolean algebra of the clopen set of X and the theory of $C(X)$.

Theorem (5.3.3). Let X and Y be compact 0-dimensional Hausdorff spaces. Then $C(X)$ and $C(Y)$ are elementarily equivalent if and only if the Boolean algebras $CL(X)$ and $CL(Y)$ are elementarily equivalent.

We obtain several corollaries of the above theorem. For example, we show that many familiar spaces have function spaces which are elementarily equivalent, and hence have isomorphic ultrapowers.

We study saturation properties in the abelian setting. We find that if $C(X)$is countably degree-1 saturated then X is a sub-Stonean space without the countable chain condition and which is not Rickart. In the 0-dimensional setting we describe the relation between the saturation of $C(X)$and the saturation of $CL(X)$. While some implications hold in general, a complete characterization occurs in the case where X has no isolated points. The following is a special case of Theorems(5.3.41) and(5.3.42).

Theorem (5.3.4). Let X be a compact 0-dimensional Hausdorff space without isolated points. Then the following are equivalent:

(a) $C(X)$ is countably degree-1 saturated,
(b) $C(X)$ is countably saturated,
(c) $CL(X)$ is countably saturated.

Before beginning the technical portion, we wish to give further illustrations of the importance of the saturation properties we will be considering, particularly the full model-theoretic notion of saturation (see Definition (5.3.8) below). For countable degree-1 saturation we refer to Theorem(5.3.14) for a list of consequences. The following fact follows directly from the fact that axiomatizable properties are preserved to ultrapowers, which are countably saturated (see [335]).

Fact(5.3.5)[331]: Let P be a property that may or may not be satisfied by a C^*-algebra. Suppose that countable saturation implies the negation of P. Then P is not axiomatizable (in the sense of [335]).

Other interesting consequences follow when the Continuum Hypothesis is also assumed. In this case, all ultrapowers of a separable algebra by a non-principal ultrafilter on ℕ are isomorphic. In fact, all that is needed is that the ultrapowers are countably saturated and elementarily equivalent:

Fact(5.3.6)[331]: (See [119].) Assume the Continuum Hypothesis. Let A and B be two elementary equivalent countably saturated C^*-algebras of density $\varkappa_1$. Then $A \cong B$.

Applying Parovicenko's Theorem (see [33]), the above fact immediately yields that under the Continuum Hypothesis if X and Y are locally compact Polish 0-dimensional spaces then $C(\beta X\backslash X) \cong C(\beta Y\backslash Y)$.

Saturation also has consequences for the structure of automorphism groups:

Fact(5.3.7)[331]: (See [347].) Assume the Continuum Hypothesis. Let A be a countably saturated C^*-algebra of density $\varkappa_1$. Then A has $2^{\varkappa_1}$-many automorphisms. In particular, A has outer automorphisms.

It is known that for Fact(5.3.7) the assumption of countable saturation can be weakened in some particular cases (see [118] and [114]), and the property of having many automorphisms under the Continuum Hypothesis is shared by many algebras that are not even quantifier-free saturated (for example the Calkin algebra). In particular it is plausible that the assumption of countable saturation in Fact(5.3.7) can be replaced with a lower degree of saturation.

In light of this, and since the consistency of the existence of non-trivial homeomorphisms (see [347]) of spaces of the form $\beta\mathbb{R}^n \setminus \mathbb{R}^n$ is still open (for $n \geq 2$), it makes sense to ask about the saturation of $C(\beta\mathbb{R}^n \setminus \mathbb{R}^n)$. In the opposite direction, the Proper Forcing Axiom has been used to show the consistency of all automorphisms of certain algebras being inner.

We begin by reviewing the definition and basic properties. Since finite-dimensional C^*-algebras have full model-theoretic saturation, and hence have all of the weakenings in which we are interested, we assume throughout that all C^*-algebras under discussion are infinite dimensional unless otherwise specified.

We will be considering C^*-algebras as structures for the continuous logic formalism of [335] (or, for the more specific case of operator algebras, [119]). For many of the results it is not necessary to be familiar with that logic. Informally, a formula is an expression obtained from a finite set of norms of $*$-polynomials with complex coefficients by applying continuous functions and taking suprema and infima over some of the variables. A formula is quantifier-free if it does not involve suprema or infima. Aformula is a sentence if every variable appears in the scope of a supremum or infimum. [119] for the precise definitions. Given a C^*-algebra A we will denote as $A_{\leq 1}, A_1$ and A^+ the closed unit ball of A, its boundary, and the cone of positive elements respectively.

Definition(5.3.8)[331]: Let A be a C^*-algebra, and let Φbe a collection of formulas in the language of C^*-algebras. We say that A is countably Φ-saturated if for every sequence $(\emptyset_n)_{n\in\mathbb{N}}$ of formulas from $\Phi$with parameters from $A_{\leq 1}$, and sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets, the following are equivalent:

(i) There is a sequence $(b_k)_{k\in\mathbb{N}}$ of elements of $A_{\leq 1}$ such that $\emptyset_n^A(\bar{b}) \in K_n$ for all $n \in \mathbb{N}$;
(ii) For every $\epsilon > 0$ and every finite $\Delta \subset \mathbb{N}$ there is $(b_k)_{k\in\mathbb{N}} \subseteq A_{\leq 1}$, depending on ϵ and A, such that $\emptyset_n^A(\bar{b}) \in (K_n)_\epsilon$ for all $n \in \Delta$.

The three most important special cases for us will be the following:

(a) If Φcontains all 1-degree $*$-polynomials, we say that A is countably 1-degree saturated.
(b) If Φcontains all quantifier-free formulas, we say that A is quantifier-free saturated.
(c) If $\Phi$is the set of all formulas we say that the algebra A is countably saturated.

Clearly condition (i) in the definition always implies condition (ii), but the converse does not always hold. We recall the (standard) terminology for the various parts of the above definition. A set of conditions satisfying (ii) in the definition is called a type*;* we say that the conditions are approximately finitely satisfiable or consistent. When condition (i) holds, we say that the type is realized (or satisfied) by $(b_k)_{k\in\mathbb{N}}$.
An equivalent definition of quantifier-free saturation is obtained by allowing only $*$-polynomials of degree at most 2 [114]. By (model-theoretic) compactness the concepts defined by Definition(5.3.8) are unchanged if each compact set K_n is assumed to be a singleton.

In the setting of logic for C^*-algebras, the analogue of a finite discrete structure is a C^*-algebra with compact unit ball, that is, a finite-dimensional algebra. The following fact is then the C^*-algebra analogue of a well-known result from discrete logic.

Fact(5.3.9)[331]: (See [335].) Every ultraproduct of C^*-algebras over a countably incomplete ultrafilter is countably saturated. In particular, every finite-dimensional C^*-algebra is countably saturated.

The second part of the fact follows from the first because any ultrapower of a finite-dimensional C^*-algebra is isomorphic to the original algebra (see [335]).

A condition very similar to the countable saturation of ultraproducts was considered by Kirchberg and Rørdam under the name "ϵ-test" in [352]. Before returning to the analysis of the different degrees of saturation, we give definitions for two well-known concepts that we are going to use strongly, but that may not be familiar to a C^*-algebraist.

Definition(5.3.10)[331]: The theory of a C^*-algebra A is the set of all sentences in the language of C^*-algebras which have value 0 when evaluated in A. We say that C^*-algebras A and B are elementary equivalent, written $A \equiv B$, if their theories are equal.

Elementary equivalence can be defined without reference to continuous logic by way of the following result, which is known as the Keisler-Shelah theorem for metric structures. The version we are using is stated in [335], and was originally proved in an equivalent setting in [350].

Theorem(5.3.11)[331]: Let A and B be C^*-algebras. Then $A \equiv B$ if and only if there is an ultrafilter $\mathcal{U}$ (over a possibly uncountable set) such that the ultrapowers $A^{\mathcal{U}}$ and $B^{\mathcal{U}}$ are isomorphic.

Definition(5.3.12)[331]: Let A be a C^*-algebra. We say that the theory of A has quantifier elimination if for any formula $\emptyset(\bar{x})$ and any $\epsilon > 0$ there is quantifier-free formula $\psi(\bar{x})$such that for every C^*-algebra B satisfying $A \equiv B$, and any $\bar{b} \subseteq B$ (of the appropriate length) we have that in B,

$$|\emptyset(\bar{b}) - \psi(\bar{b})| \leq \epsilon.$$

Countable degree-1 saturation is the weakest form of saturation that we will consider. Even this modest degree of saturation for a C^*-algebra has interesting consequences. In particular it implies several properties (see the detailed definition before) that were shown to hold in coronas of σ-unital algebras in [34]

Definition(5.3.13)[331]: (See [34].) Let A be a C^*-algebra. Then A is said to be

(a) SAW^* if any two σ-unital subalgebras C, B are orthogonal (i.e., $bc = 0$ for all $b \in B$ and $c \in C$) if and only if are separated (i.e., there is $x \in A$ such that $xbx = b$ for all $b \in B$ and $xc = 0$ for all $x \in C$);

(b) AA-CRISP if for any sequences of positive elements $(a_n), (\mathrm{b_n})$such that for all n we have $a_n < a_{n+1} \leq \ldots \leq b_{n+1} < b_n$ and any separable $D \subseteq A$ such that for all $d \in D$ we have $\lim_n \|[d, a_n]\| = 0$, there is $c \in A^+$ such that $a_n \leq c \leq b_n$ for any n and for all $d \in D$ we have $[c, d] = 0$;

(c) σ-sub-Stonean if whenever $C \subseteq A$ is separable and $a, b \in A^+$ are such that $aCb = \{0\}$ then there are contractions $f, g \in \mathrm{C}' \cap A$ such that $fg = 0, f_a = a$ and $gb = \mathrm{b}, \mathrm{C}' \cap \mathrm{A}$ denoting the relative commutant of C inside A.

Theorem (5.3.14)[331]: (See [114].) Let A be a countably degree-1 saturated C^*-algebra. Then:
(a) A is SAW^*,

(b) A is AA-CRISP,

(c) A satisfies the conclusion of Kasparov's technical theorem (see [20]),

(d) A is a-sub-Stonean,

(e) every derivation of a separable subalgebra of A is of the form $\delta_b(x) = bx - xb$ for some $b \in A$,

(f) A is not the tensor product of two infinite dimensional C^*-algebras (this is a consequence of being SAW^*).

It is useful to know that when a degree-1 type can be approximately finitely satisfied by elements of a certain kind then the type can be realized by elements of the same kind.
Lemma(5.3.15)[331]: (See [114].) Let A be a countably degree-1 saturated C^*-algebra. If a type can be finitely approximately satisfied by self-adjoint elements then it can be realized by self-adjoint elements, and similarly with "self-adjoint" replaced by "positive".
We will also make use of the converse of the preceding lemma, which says that to check countable degree-1 saturation it is sufficient to check that types which are approximately finitely satisfiable by positive elements are realized by positive elements.
Lemma(5.3.16)[331]: Suppose that A is a C^*-algebra that is not countably degree-1 saturated. Then there is a countable degree-1 type which is approximately finitely satisfiable by positive elements of A but is not realized by any positive element of A.
Proof: Let $(P_n(\bar{x}))_{n\in\mathbb{N}}$ be degree-1 polynomials, and $(K_n)_{n\in\mathbb{N}}$ compact sets, such that the type $\{||P_n(\bar{x})|| \in K_n : n \in \mathbb{N}\}$ is approximately finitely satisfiable but not satisfiable in A. For each variable x_k, we introduce new variables v_k, w_k, y_k, and z_k. For each n, let $Q_n(\bar{v},\bar{w},\bar{y},\bar{z})be$ the polynomial obtained by replacing each x_k in P_n by $v_k + iw_k - y_k - iz_k$. Since every $x \in A$ can be written as $x = v + iw - y - iz$ where $v, w, y, z \in A^+$, it follows that $\{\|Q_n(\bar{v},\bar{w},\bar{y},\bar{z})\| \in K_n : n \in \mathbb{N}\}$ is approximately finitely satisfiable (respectively, satisfiable) by positive elements in A if and only if $\{\|P_n(\bar{x})\| \in K_n : n \in \mathbb{N}\}$ is approximately finitely satisfiable (respectively, satisfiable).

The first example of an algebra which fails to be countably degree-1 saturated is $B(H)$, where H is an infinite dimensional separable Hilbert space. In fact, no infinite dimensional separable C^*-algebra can be countably degree-1 saturated; this was observed in [114]. We include here a proof of the slightly stronger result, enlarging the class of algebras that are not countably degree-1 saturated.

Definition(5.3.17)[331]: A C^*-algebra A has few orthogonal positive elements if every family of pairwise orthogonal positive elements of A of norm 1is countable.

We will further examine the property of having few orthogonal positive elements in the context of abelian C^*-algebras. For now, we have the following lemma:

Lemma(5.3.18)[331]: If an infinite dimensional C^*-algebra A has few orthogonal positive elements, then A is not countably degree-1 saturated.

Proof: Suppose to the contrary that A has few orthogonal positive elements and is countably degree-1 saturated. Using Zorn's lemma, find a set $Z \subseteq A_1^+$ which is maximal (under inclusion) with respect to the property that if $x, y \in Z$ and $x \neq y$, then $xy = 0$. By hypothesis, the set Z is countable; list it as $Z = \{a_n\}_{n\in\mathbb{N}}$.
For each $n \in \mathbb{N}$, define $P_n(x) = a_n x$, and let $K_n = \{0\}$. Let $P_{-1}(x) = x$, and $K_{-1} = \{1\}$. The type $\{\|P_n(x)\| \in K_n : n \geq -1\}$ is finitely satisfiable. Indeed, by definition of Z, for any $m \in \mathbb{N}$ and any $0 \leq n \leq m$ we have $\|P_n(a_{m+1})\| = \|a_n a_{m+1}\| = 0$, and $\|a_{m+1}\| = 1$. By countable degree-1 saturation and Lemma(5.3.15) there is a positive element $b \in A_1^+$ such that $\|P_n(b)\| = 0$ for all $n \in \mathbb{N}$. This contradicts the maximality of Z.

Subalgebras of $\mathfrak{B}(H)$ clearly have few positive orthogonal elements, whenever H is separable. As a result, we obtain the following.

Corollary(5.3.19)[331]: No infinite dimensional subalgebra of $\mathfrak{B}(H)$, with H separable, can be countably degree-1 saturated.

Corollary(5.3.19) shows that many familiar C^*-algebras fail to be countably degree-1 saturated. In particular, it implies that no infinite dimensional separable C^*-algebra is countable degree-1 saturated. Corollary (5.3.19) also shows that the class of countably degree-1 saturated algebras is not closed under taking inductive limits (consider, for example, the CAR algebra $\bigotimes_{i=1}^{\infty} M_2(\mathbb{C})$, or any AF algebra) or subalgebras. On the other hand, several examples of countably degree-1 saturated algebras are known. It was shown in [114] that every corona of a σ-unital algebra is countably degree-1 saturated. Recently Voiculescu in [37] found examples of algebras which are not C^*-algebras, but which have the unexpected property that their coronas are countably degree-1 saturated C^*-algebras. The results of the following expand the list of examples of countably degree-1 saturated C^*-algebras.

We recall some definitions which we will need:

Definition(5.3.20)[331]: A C^*-algebra A is σ-unital (see [100]) if it has a countable approximate identity, that is, a sequence $(e_n)_{n\in\mathbb{N}}$ such that for all $x \in A$,

$$\lim_{n\to\infty} \|e_n x - x\| = \lim_{n\to\infty} \|x e_n - x\| = 0.$$

A closed ideal $I \subseteq A$ is essential (see [100]) if it has trivial annihilator, that is, if $\{x \in A : I_x = \{0\}\} = \{0\}$.

Notation(5.3.21)[331]: Let $\mathcal{R}$ be the hyperfinite II_I factor. Let $M = \mathcal{R} \bar{\otimes} \mathfrak{B}(H)$ be the unique hyperfinite II_∞ factor associated to $\mathcal{R}$, and let $\mathcal{T}$ be its unique trace. We denote by K_m the unique norm closed two-sided ideal generated by the positive elements of finite trace in M.

Note that M is the multiplier algebra of $\mathcal{K}_M$, so the quotient $M/\mathcal{K}_M$ is the corona of M.

Any ideal in a von Neumann algebra is generated, as a linear space, by its projections, hence $\mathcal{K}_M$ is the closure of the linear span in M of the set of projections of finite trace. In particular, $\mathcal{R} \otimes \mathcal{K}(H) \subsetneq \mathcal{K}_M$. To see that the inclusion is proper, fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for H, and choose $(p_n)_{n\in\mathbb{N}}$from $\mathcal{R}$ such that $\mathcal{T}(p_n) = 2^{-\mathrm{n}}$ for all $n \in \mathbb{N}$. For each n, let $qn \in \mathfrak{B}(\mathrm{H})$ be the projection onto e_n, and let $q = \sum_{\mathrm{n}} p_n \otimes q_n$. Then $q \in M$ is a projection of finite trace, but $q \notin \mathcal{R} \otimes \mathcal{K}(H)$.

We recall few well known properties of this object.

Proposition(5.3.22)[331]: (See [35].)

(i). $\mathcal{R} \cong M_p(\mathcal{R})$ for every prime number p. Consequently $M_n(\mathcal{R} \cong M_m(\mathcal{R})$ for every $m, n \in \mathbb{N}$.

(ii). $K_0(\mathcal{K}_m) = \mathbb{R} = K_1(M/\mathcal{K}_m)$.

(iii) $\mathcal{K}_m$ is not σ-unital.

(iv). $\mathcal{K}_m \otimes \mathcal{K}(H)$ is not isomorphic to $\mathcal{R} \otimes \mathcal{K}(H)$.

Proof: (i). This is because $M_p(\mathcal{R})$is hyperfinite and $\mathcal{R}$ is the unique hyperfinite II_1-factor. (ii). Note that $K_0(M) = 0 = K_1(M)$and apply the exactness of the six term K-sequence. (iii). Suppose to the contrary that$(x_n)_{n\in\mathbb{N}}$ is a countable approximate identity in $\mathcal{K}_M$formed by positive elements such that $0 \leq x_n \leq 1$ for all n. Using spectral theory, we can find projections $p_n \in \mathcal{K}_M$ such that $\|p_n x_n - x_n\| \leq 1/n$ for each n. Then $(p_n)_{n\in\mathbb{N}}$ is again a countable approximate identity for $\mathcal{K}_M$. For each $n \in \mathbb{N}$ define $q_n = \sup_{\mathrm{k}\leq\mathrm{n}} pk \in \mathcal{K}_M$, and by passing to a subsequence we can suppose that $(q_n)_{n\in\mathbb{N}}$ is strictly increasing. For each $n \in \mathbb{N}$ find a projection $r_n < \mathrm{q}_{\mathrm{n}+1} - q_n$ such that $\mathcal{T}(r_n) \leq \frac{1}{2^{\mathrm{n}}}$. Then $r = \sum_{n\in\mathbb{N}} r_n \in \mathcal{K}_{\mathrm{M}}$, and we have that for all $n \in \mathbb{N}$,

$$\|q_n r - r\| = 1.$$

This contradicts that $(q_n)_{n\in\mathbb{N}}$ is an approximate identity.

(iv). This follows from (iii), since $\mathcal{R}\otimes\mathcal{K}(H)$has a countable approximate identity and $\mathcal{K}_M\otimes\mathcal{K}(H)$does not. To see this, suppose that $(x_n)_{n\in\mathbb{N}}$ is a countable approximate identity for $\mathcal{K}_M\otimes\mathcal{K}(H)$, and let p be a rank one projection in $K(H)$.

Then $((1 \otimes p)_{x_n}(1 \otimes p))_{n\in\mathbb{N}}$ is a countable approximate identity for $K_m \otimes p$, but $K_m \otimes p \cong \mathcal{K}_{\mathrm{M}}$, so this contradicts (iii).

There are many differences between the Calkin algebra and $M/\mathcal{K}_M$. Some of them are already clear from the K -theory considerations above, or from the fact that $\mathcal{K}(H)$is separable. Another difference, a little bit more subtle, is given by the following:

Proposition(5.3.23)[331]: Let H be a separable Hilbert space, and let Q be the canonical quotient map onto the Calkin algebra. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for H, and let $S \in \mathrm{B(H)}$ be the unilateral shift in $\mathfrak{B}(H)$ defined by $S(e_n) = \mathrm{e}_{\mathrm{n}+1}$ for all n. Then neither S nor $Q(S)$ has a square root, but $1 \otimes S \in \mathcal{R}\bar{\otimes}B(H)$ does have a square root.

Proof: Suppose that $Q(T) \in C(H)$is such that $Q(T)^2 = Q(S)$. Since $Q(S)$is invertible in the Calkin algebra so is $Q(T)$. The Fredholm index of S is -1, so if $n \in \mathbb{Z}$ is the Fredholm index of T then $2n = -1$, which is impossible. Therefore $Q(S)$has no square root, and hence neither does S.

For the second assertion recall that $R \cong M_2(R)$, and so

$$\mathcal{R}\bar{\otimes}\mathfrak{B}(H) \cong M_2(\mathcal{R}\bar{\otimes}\mathfrak{B}(H) = \mathcal{R}\bar{\otimes}\ (M_2 \otimes\mathfrak{B}(H)).$$

We view $\mathfrak{B}(H)$ as embedded in $M_2 \otimes\mathfrak{B}(\mathrm{H}) = \mathfrak{B}(\mathrm{H}')$ for another Hilbert space H'; find $(f_n)_{n\in\mathbb{N}}$ such that $\{ e_n, f_n : n \in \mathbb{N} \}$ is an orthonormal basis for H'. Let $S' \in \mathfrak{B}(\mathrm{H}')$ be defined such that $S'(e_n) = f_n$ and $S'(f_n) = e_{n+1}$ for all n. Then $T = 1 \otimes S' \in \mathrm{R}\bar{\otimes}\mathrm{S}(\mathrm{H}')$, and $T^2 = 1 \otimes \mathrm{S}$.

A consequence of the previous proof, and of the fact that $\mathcal{R} \cong M_p(\mathcal{R})$ for any integer p, is the following:

Corollary(5.3.24)[331]: $1\otimes S \in M$ has a qth-root for every rational q.

With the motivating example in mind, we turn to establishing countable degree-1 saturation of a class of algebras containing $/\mathcal{K}_M$. .

We recall the following result, which may be found in [34]:

Lemma(5.3.25)[331]: Let A be a C^*-algebra, $S \in A_1$ and $T \in A_1^+$. Then

$$\|S, T\| = \epsilon \leq \frac{1}{4} \Rightarrow \|S, T^{1/2}\| \leq \frac{5}{4}\sqrt{\epsilon}.$$

The following lemma is the key technical ingredient of Theorem(5.3.27) below. It is a strengthening of the construction used in [114], as if A is σ-unital and $M = M(A)$ is the multiplier algebra of A, then M and A satisfy the hypothesis of our lemma.

Lemma(5.3.26)[331]: Let M be a unital C^*-algebra, let $A \subseteq M$ be an essential ideal, and let $\pi: M \longrightarrow M/A$ be the quotient map. Suppose that there is an increasing sequence $(g_n)_{n\in\mathbb{N}} \subset A$ of positive elements whose supremum is 1_M, and suppose that any increasing uniformly bounded sequence converges in M.

Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of the unit ball of M and $(\epsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to 0, with $\epsilon_0 < \frac{1}{4}$. Then there is an increasing sequence $(e_n)_{n\in\mathbb{N}} \subset A_{\leq 1}^+$ such that, for all $n \in \mathbb{N}$ and $a \in F_n$, the following conditions hold, where $f_n = (e_{n+1} - e_n)^{1/2}$:

(i) $|\|(1 - e_{n-2})a(1 - e_{n-2})\| - \|\pi(a)\|| < \epsilon_n for\ all\ n \geq 2$

(ii) $\|[f_n, a]\| < \epsilon_n\ for\ all\ n,$

(iii) $\|f_n(1 - e_{n-2}) - f_n\| < \epsilon_n\ for\ all\ n \geq 2,$

(iv) $\|f_n f_m\| < \epsilon_m\ for\ all\ m \geq n + 2,$

(v) $\|[f_n, f_{n+1}]\| < \epsilon_{n+1}\ for\ all\ n,$

(vi) $\|f_n a f_n\| \geq \|\pi(a)\| - \epsilon_n\ for\ all\ n,$

(vii) $\sum_{n\in\mathbb{N}} f_n^2 = 1;$

and further, whenever $(e_n)_{n\in\mathbb{N}}$ is a bounded sequence from M, the following conditions also hold:

(viii) the series $\sum_{n\in\mathbb{N}} f_n x_n f_n$ converges to an element of M,
(ix) $\|\sum_{n\in\mathbb{N}} f_n x_n f_n\| \leq \operatorname{SUP}_{n\in\mathbb{N}}\|x_n\|$
(x) whenever $\limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty}\|x_n f_n^2\|$ we have

$$\limsup_{n\to\infty}\|x_n f_n^2\| \leq \left\|\pi\left(\sum_{n\in\mathbb{N}} x_n f_n^2\right)\right\|.$$

Proof: For each $n \in \mathbb{N}$ let $\delta_n = 10^{-100}$ and let $(g_n)_{n\in\mathbb{N}}$ be an increasing sequence in A whose weak limit is 1. We will build a sequence $(e_n)_{n\in\mathbb{N}}$ satisfying the following conditions:
(a)$|\|(1 - e_{n-2})a(1 - e_{n-2})\| - \|\pi(a)\|| < \epsilon_n$ for all $n \geq 2$and $a \in F_n$,
(b)$0 \leq e_0 \leq \ldots \leq e_n \leq e_{n+1} \leq \cdots \leq 1$, and for all n we have $e_n \in A$,
(c) $\|e_n e_k - e_k\| < \delta_{n+1}$ for all $n > k$,
(d)$\|[e_n, a]\| < \delta_n$ for all $n \in \mathbb{N}$ and $a \in F_{n+1}$,
(e)$\|(e_{n+1} - e_n)a\| \geq \|\pi(a)\| - \delta_n$ for all $n \in \mathbb{N}$ and $a \in F_n$,
(f) $\|(e_{m+1} - e_m)^{1/2} e_n (e_{m+1} - e_m)^{1/2} - (e_{m+1} - e_m)\| < \delta_{n+1}$ for all $n > m + 1$,
(g)$e_{n+1} \geq g_{n+1}$ for all $n \in \mathbb{N}$.
We claim that such a sequence will satisfy (i)-(vii), in light of Lemma (5.3.26). Conditions (i) and (a) are identical. Condition (d) implies condition (ii). Condition (c) and the C^*-identity imply condition (iii), which in turn implies conditions (iv) and (v). We have also that conditions (e) and (g) imply respectively conditions (vi) and (vii), so the claim is proved. After the construction we will show that (viii)-(x) also hold.
Take $\Lambda = (\lambda \in A^+ : \lambda \leq 1\}$ to be the approximate identity of positive contractions (indexed by itself) and let Λ' be a subnet of Λ that is quasicentral for M (see [34] or [7]).
Since A is an essential ideal of M, by [100] there is a faithful representation β on a Hilbert space H such that

$$1_H = \mathrm{SOT} - \lim_{\lambda\in\Lambda'}\{\beta(\lambda)\}.$$

Consequently, for every finite $F \subset M, \epsilon > 0$ and $\lambda \in \Lambda'$ there is $\mu > A$ such that for all $a \in F$,
$v \geq \mu \Rightarrow \|(v - \lambda)a\| \geq \|\pi(a)\| - \epsilon.$

We will proceed by induction. Let $e_{-1} = 0$ and $\lambda_0 \in \Lambda'$ be such that for all $\mu > \lambda_0$ and $a \in F_1$ we have $\|[\mu, a]\| < \delta_0$. By cofinality of Λ' in Λ we can find an $e_0 \in \Lambda'$ such that $e_0 > \lambda_0$, g_0. Find now $\lambda_1 > e_0$ such that for all $\mu > \lambda_1$ and $a \in F_2$ we have

$$\|[\mu, a]\| < \delta_1, \|\mu - e_0)a\| \geq \|\pi(a)\| - \delta_1.$$

Since we have that

$$\|\pi(a)\| = \lim_{\lambda\in\Lambda'}\|(1 - \lambda)a(1 - \lambda)\|$$

we can also ensure that for all $a \in F_3$ and all $\mu > \lambda_1$, condition (i) is satisfied. Picking $e_1 \in \Lambda'$ such that $e_1 > \lambda_1, g_1$ we have that the base step is completed. Suppose now that $e_0, \ldots, e_n, f_0, \ldots, f_{n-1}$ are constructed.

We can choose $\curlyvee_{n+1}$ so that for all $\mu >\curlyvee_{n+1}$, with $\mu \in \Lambda'$, we have $\|[\mu, a]\| < \delta_{n+1}/4$ and $\|(\mu - e_n)a\| \geq \|\pi(a)\| - \delta_n$ for $a \in F_{n+2}$. Moreover, by the fact that Λ' is an approximate identity for A we can have that $\|f_m\mu fm - f_m^2\| < \delta_{n+2}$ for every $m < n$ and that $\|\mu e_k - e_k\| < \delta_{n+2}$ for all $k \leq$ n. By Eq. (i) we can also ensure that for all $a \in F_{n+2}$ and all $\mu >\curlyvee_{n+1}$, condition (1) is satisfied.

Once this $\curlyvee_{n+1}$ is picked we may choose

$$e_{n+1} \in \Lambda', e_{n+1} >\curlyvee_{n+1}, g_{n+1},$$

to end the induction.

It is immediate from the construction that the sequence $(e_n)_{n\in\mathbb{N}}$ chosen in this way satisfies conditions (a)-(g). To complete the proof of the lemma we need to show that conditions (viii), (ix) and (x) are satisfied by the sequence $\{f_n\}$.

To prove (viii), we may assume without loss of generality that each x_n is a contraction. Recall that every contraction in M is a linear combination (with complex coefficients of norm 1) of four positive elements of norm less than 1, and addition and multiplication by scalar are weak operator continuous functions. It is therefore sufficient to consider a sequence $(x_n)$of positive contractions. By positivity of x_n, we have that $(\sum_{i\leq n} f_i x_i f_i)_{n\in\mathbb{N}}$ is an increasing uniformly bounded sequence, since for every n we have

$$\sum_{i\leq n} f_i x_i f_i \sum_{i\leq n} f_i^2 \quad and \quad f_n x_n f_n \geq 0.$$

Hence $(\sum_{i\leq n} f_i x_i f_i)_{n\in\mathbb{N}}$ converges in weak operator topology to an element of M of bounded norm, namely the supremum of the sequence, which is $\sum_{n<\mathbb{N}} f_n x_n f_n$.

For (ix), consider the algebra $\prod_{k\in\mathbb{N}} M$ with the sup norm and the map $\emptyset_n\colon \prod_{k\in\mathbb{N}} M \longrightarrow M$ such that $\emptyset_n\big((x_i)\big) = f_n x_n f_n$. Each $\emptyset_n$ is completely positive, and since $f_n^2 \leq \sum_{i\in\mathbb{N}} f_i^2 = 1$, also contractive. For the same reason the maps $\psi_n\colon \prod_{k\in\mathbb{N}} M \longrightarrow M$ defined as $\psi_n((x_i)) = \sum_{j\leq n} f_j x_j f_j$ are completely positive and contractive. Take Ψ to be the supremum of the maps ψ_n. Then $\Psi\big((x_n)\big) = \sum_{j\leq\mathbb{N}} f_i x_i f_i$. This map is a completely positive map of norm 1, because $\|\Psi\| = \|\Psi(1)\|$and from this condition (ix) follows.

For (x), we can suppose $\operatorname{limsup}_{i\to\infty} \|x_i\| = \operatorname{limsup}_{i\to\infty} \left\|x_i f_i^2\right\| = 1$. Then for all $\epsilon > 0$ there is a sufficiently large $m \in \mathbb{N}$ and a unit vector $\xi_m \in H$ such that .

$$\|x_m f_m^2(\xi_m)\| \geq 1 - \epsilon.$$

Since $||x_i|| \leq 1$ for all i, we have that $\left||f_m(\xi_m)|\right| \geq 1 - \epsilon$, that is, $|(f_m^2 \xi_m \mid \xi_m)| \geq 1 - \epsilon$. In particular we have that$\|\xi_m - f_m^2(\xi_m)\| \leq \epsilon$.

Since $\sum f_i^2 = 1$ we have that ξ_m and $\xi_n$constructed in this way are almost orthogonal for all n, m. In particular, choosing ϵ small enough at every step, we are able to construct a sequence of unit vectors $\{\xi_m\}$ such that $|(\xi_m \mid \xi_n)| \leq 1/2^m$ for $m > n$. But this means that for any finite projection $P \in M$ only finitely many ξ_m are in the range of P up to ϵ for every $\epsilon > 0$. In particular, if I is the set of all convex combinations of finite projections, we have that

$$\lim_{\curlyvee\in I} \left\| \sum_{i\in\mathbb{N}} x_i f_i^2 - \curlyvee \left(\sum_{i\in\mathbb{N}} x_i f_i^2 \right) \right\| \geq 1.$$

Since I is an approximate identity for A we have that

$$\left\|\pi\sum_{i\in\mathbb{N}} x_i f_i^2\right\| = \lim_{\curlywedge\in I}\left\|\sum_{i\in\mathbb{N}} x_i f_i^2 - \curlywedge\left(\sum_{i\in\mathbb{N}} x_i f_i^2\right)\right\|,$$

as desired.

We can then proceed with the proof of the fellowing theorem.

Theorem(5.3.27)[331]: Let M be a unital C^*-algebra, and let $A \subseteq M$ be an essential ideal. Suppose that there is an increasing sequence $(g_n)_{n\in\mathbb{N}} \subset A$ of positive elements whose supremum is 1_M, and suppose that any increasing uniformly bounded sequence converges in M. Then M/A is countably degree-1 saturated.

Proof: Let $\pi: M \longrightarrow M/A$ be the quotient map. Let $(P_n(\bar{x}))_{n\in\mathbb{N}}$ be a collection of $*$ polynomial of degree 1with coefficients in M/A, and for each $n \in \mathbb{N}$ let $r_n \in \mathbb{R}^+$. Without loss of generality, reordering the polynomials and eventually adding redundancy if necessary, we can suppose that the only variables occurring in P_n are $x_0, \ldots, x_n$.

Suppose that the set of conditions $\{\|P_n(x_o, \ldots, x_n)\| = r_n : n \in \mathbb{N}\}$ is approximately finitely satisfiable, in the sense of Definition(5.3.8). As we noted immediately after Definition(5.3.8), it is sufficient to assume that the partial solutions are all in $(M/A)_{\leq 1}$, and we must find a total solution also in $(M/A)_{\leq 1}$. So we have partial solutions

$$\{\pi(x_{k,i})\}_{k\leq i} \subseteq (M/A)_{\leq 1}$$

such that for all $i \in \mathbb{N}$ and $n \leq i$ we have

$$\left\|P_n\{\pi(x_{0,i}), \ldots, \pi(x_{n,i}))\right\| \in (r_n)_{1/i}.$$

For each $n \in \mathbb{N}$, let $Q_n(x_0, \ldots, x_0)$. be polynomial whose coefficients are lifting s of the coefficients of P_n to M, and let F_n be a finite set that contains

(d) all the coefficients of Q_k, for $k \leq n$;
(e) $x_{k,i}, x_{k,i}^*$ for $k \leq i \leq n$;
(f) $Q_k(\mathrm{x}_0, \mathrm{i}, \ldots, x_{k,i})$for $\mathrm{k} \leq i \leq n$.

Let $\epsilon_n = 4^{-\mathrm{n}}$. Find sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ satisfying the conclusion of Lemma(5.3.26) for these choices of $(f_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$.
Let $\bar{x}_{n,i} = (\mathrm{x}_{\mathrm{o,i}}, \ldots, x_{n,i}), yk = \sum_{i\geq k} f_i x_{k,i} f_i$, $\bar{y}_n = (y_0, \ldots, y_n)$ and $\bar{Z}_n = \pi(\bar{y}_n)$. Fix $n \in \mathbb{N}$; we will prove that $\|P_n(\bar{Z}_n)\| = r_n$.
First, since $x_{k,i} \in M_{\leq 1}$, as a consequence of condition (ix) of Lemma (5.3.26), we have that $y_i \in M_{\leq 1}$ for all i. Moreover, since Q_n is a polynomial whose coefficients are lifting of those of P_n we have

$$\|P_n(\bar{Z}_n)\| = \|\pi(Q_n(\bar{y}_n))\|.$$

We claim that

$$Q_n(\bar{y}_n) - \sum_{j\in\mathbb{N}} f_j\, Q_n(\bar{x}_{n,j}) f_j \in A\,.$$

It is enough to show that

$$\sum_{j\in\mathbb{N}} f_j\, ax_{k,j}\, bf_j \; - \sum_{j\in\mathbb{N}} af_j\, x_{\,k,j} f_j b \;\in A\,,$$

where a, b are coefficients of a monomial in Q_n, since Q_n is the sum of finitely many of these elements (and the proof for monomials of the form $ax^*_{k,j}b$ is essentially the same as the one for $ax_{k,j}b$).

By construction we have $a, b \in F_n$, and hence by condition (ii) of Lemma(5.3.26), for j sufficiently large,

$$\forall x \in M_{\leq i}\ (\left\|af_j x f_j b - f_j\, ax\, bfj\right\| \leq 2^{-j}(\|a\| + \|b\|)).$$

Therefore $\sum_{j\in\mathbb{N}}(f_j\, ax_{k,j}\, bf_j - af_j\, x_{\,k,j} f_j b)$ is a series of elements in A that is converging in norm, which implies that the claim is satisfied. In particular,

$$\|P_n(\bar{Z}_n)\| = \left\|\pi\left(\sum_{j\in\mathbb{N}} f_j Q_n(\bar{x}_{n,j}) f_j\right)\right\|$$

For each $j \geq 2$, let $a_j = (1-e_{j-2})Q_n(\bar{x}_{n,j})(1-\mathrm{e}_{\mathrm{j}-2})$. By condition (i) of Lemma(5.3.26), the fact that $Q_n(\bar{x}_{n,j}) \in F_n$, and the original choice of the $x_{n,j}$'s, we have that limsup $\left\|a_j\right\| = r_n$. Similarly to the above, but this time using condition (iii) of Lemma (5.3.26), we have

$$\left\|\pi\left(\sum_{j\in\mathbb{N}} f_j Q_n(\bar{x}_{n,j}) f_j\right)\right\| = \left\|\sum_{j\in\mathbb{N}} f_j a_j f_j\right\| \leq \left\|\sum_{j\in\mathbb{N}} f_j a_j f_j\right\|.$$

Using condition (ix) of Lemma(5.3.26) and the fact that $Q_n(\bar{x}_{n,j}) \in F_j$ we have that

$$\left\|\sum_{j\in\mathbb{N}} f_j a_j f_j\right\| \leq \limsup_{j\to\infty}\left\|a_j\right\| = r_n.$$

Combining the calculations so far, we have shown

$$\|P_n(\bar{Z}_n)\| = \left\|\pi\left(\sum_{j\in\mathbb{N}} f_j Q_n(\bar{x}_{n,j}) f_j\right)\right\| = \left\|\pi\left(\sum_{j\in\mathbb{N}} f_j a_j f_j\right)\right\| \leq r_n.$$

Since $Q_n(\bar{x}_{n,j}) \in F_j$ for all j, condition (vi) of Lemma (5.3.26) implies $r_n \leq \limsup_{j\to\infty}\left\|f_j Q_n(\bar{x}_{n,j}) f_j\ \right\|$.

It now remains to prove that

$$\limsup_{j\to\infty}\left\|f_j a_j f_j\right\| \leq \left\|\pi\left(\sum_{j\in\mathbb{N}} f_j a_j f_j\right)\right\|.$$

so that we will have

$$r_n \le \lim_{j\to\infty}\operatorname{Sup}\left\| f_j Q_n(\bar{x}_{n,j}) f_j \right\| = \lim_{j\to\infty}\operatorname{Sup}\left\| f_j a_j f_j \right\| \le \left\| \pi\left(\sum_{j\in\mathbb{N}} f_j a_j f_j\right)\right\| = \|P_n(\overline{z_n})\|.$$

We have $Q_n(\bar{x}_{n,j}) \in F_j$, so by condition (ii) of Lemma (5.3.26), we have that

$$\lim_{j\to\infty}\operatorname{Sup}\left\| f_j a_j f_j \right\| = \lim_{j\to\infty}\operatorname{Sup}\left\| a_j f_j^2 \right\|.$$

and hence

$$\sum_{j\in\mathbb{N}} f_j a_j f_j - \sum_{j\in\mathbb{N}} a_j f_j^2 \in A.$$

The final required claim will then follow by condition (x) of Lemma(5.3.26), once we verify

$$\lim_{j\to\infty}\operatorname{Sup}\left\| a_j f_j^2 \right\| = \lim_{j\to\infty}\operatorname{Sup}\left\| a_j \right\|$$

We clearly have that for all j,

$$\left\| a_j f_j^2 \right\| \le \left\| a_j \right\|.$$

On the other hand,

$$\begin{aligned} \lim_{j\to\infty}\operatorname{Sup}\left\| a_j f_j^2 \right\| &= \lim_{j\to\infty}\operatorname{Sup}\left\| f_j a_j f_j \right\| \\ &= \lim_{j\to\infty}\operatorname{Sup}\left\| f_j Q_n(\bar{x}_{n,j}) f_j \right\| \quad \text{by condition (iii)} \\ &> r_n = \lim_{j\to\infty}\operatorname{Sup}\left\| a_j \right\| \end{aligned}$$

The Theorem (5.3.1)bove applies, in particular, to coronas of σ-unital algebras. The following result is due to Farah and Hart, but unfortunately their proof in [114] has a technical error. Specifically, our proof of Theorem(5.3.27) uses the same strategy as in [114], but avoids their equation (10), which is incorrect.

Corollary(5.3.28)[331]: (See [114]) If A is a σ-unital c^*-algebra, then its corona $C(A)$ is countably degree-1 saturated.

We also obtain countable degree-1 saturation for the motivating example from the beginning.

Corollary(5.3.29)[331]: Let N be a II_1 factor, H a separable Hilbert space and $M = N\,\overline{\otimes} B(H)$ be the associated II_∞ factor. Let $\mathcal{K}_M$ be the unique two-sided closed ideal of M, that is the closure of the elements of finite trace. Then $M/\mathcal{K}_M$ is countably degree-1 saturated. In particular, this is the case when $N = \mathcal{R}$, the hyperfinite II_1 factor.

More generally, recall that a von Neumann algebra M is finite if there is not a projection that is Murray-von Neumann equivalent to 1_{M}, and σ-finite if there is a sequence of finite projections weakly converging to 1_{M}.

Corollary(5.3.30)[331]: Let M be a σ-finite but not finite tracial von Neumann algebra, and let A be the ideal generated by the finite trace projections. Then M/A is countably degree-1 saturated.

When H is separable, the ideal of compact operators in $B(H)$is separable, and in particular σ-unital, so it follows from Corollary(5.3.28) that the Calkin algebra is countably degree-1 saturated.

We are going to give explicit results on the theories of the generalized Calkin algebras. It is known (see [114]) that the Calkin algebra is not countably quantifier-free saturated; we show that the generalized Calkin algebras also fail to have this degree of saturation. This follows immediately from the fact that the Calkin algebra is isomorphic to a corner of the generalized Calkin algebra and that if A is a C^*-algebra that is Φ-saturated, where Φ include all C^*-polynomials of degree 1, then every corner of A is Φ -saturated. On the other hand, the proof shown below is direct and much easier than the promof in the separable case. It is worth noting, however, that the method we will use does not apply to the Calkin algebra C_0 itself.

Lemma(5.3.31)[331]: Let $\alpha \geq 1$ be an ordinal. Then C_α is not countably quantifier-free saturated.

Proof: Fix $\{A_n\}_{n\in\mathbb{N}}$ a countable partition of $\aleph_\alpha$ in disjoint pieces of size $\aleph_\alpha$ and a base $(e_\beta)_{\beta<\aleph_\alpha}$ for $H_{\aleph_\alpha}$. For each $n \in \mathbb{N}$ let P_n be the projection onto $\overline{\text{span}(e_\beta : \beta \in A_n)}$.

Claim (5.3.32)[331]: If Q is a projection in B_α such that $QP_n \in \mathcal{K}_\alpha$ for all n then Q has range of countable density.

Proof: We have that for any $n \in \mathbb{N}$ and $\epsilon > 0$ there is a finite $C_{\epsilon,n} \subset \aleph_\alpha$such that

$$\beta \notin C_{\epsilon,n} \Rightarrow \|QP_n e_\beta\| < \epsilon.$$

Let $D = \bigcup_{n\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} C_{1/m,\text{n}}$. If $\beta \notin D$ then for all $n \in \mathbb{N}$ we have $\|QP_n e_\beta\| = 0$ and since there is n such that $e_\beta \in P_n$, we have that $\|Qe_\beta\| = 0$. Since D is countable, Q is identically zero on a subspace of countable codimension.

Let $Q^{-4} = xx^* - 1, Q_{-3} = x^*x - y, Q_{-2} = y - y^* - Q_{-1} = y - y^2,$ and $Q_n = \text{yP}_\text{n}$. The type $\{\|Q_i\| = 0\}_{-4\leq i}$ admits a partial solution, but not a total solution.

We are going to have a further look at the theories of C_α. We want to see if it is possible to distinguish between the theories of C_α and of C_β, whenever $\alpha \neq \beta$. Of course, since there are at most $2^{\aleph_0}$ many possible theories, we have that there are ordinals $\alpha \neq \beta$ such that $C_\alpha = C_\beta$. As we show in the next theorem, this phenomenon cannot occur whenever α and β are sufficiently small, and similarly for B_α and B_β.

Theorem(5.3.33)[331]: Let $\alpha \neq \beta$ be ordinals, and H_α the Hilbert space of density $\aleph_\alpha$. Then the rojections of the algebras C_α and C_β as posets with respect to the Murray-von Neumann order are elementary equivalent if and only if $\alpha = \beta$ mod ω^ω, where ω^ω is computed by ordinal exponentiation, as they are the infinite projections of C_a of C_β. Consequently, if $\alpha \not\equiv \beta$ then $B_\alpha \not\equiv B_\beta$ and $C_\alpha \not\equiv C_\beta$.

Proof: The key fact is that $\alpha \not\equiv \beta$ (as first-order structures with only the ordering) if and only if $\alpha = \beta$ mod ; see [342]. Hence the proof will be complete as soon as we notice that the ordinal α is interpretable in both C_α (as the set of projections under Murray-von Neumann equivalence) and inside B_α (as the set of infinite projections under Murray-von Neumann equivalence).

There is a formula $\emptyset$ such that $\emptyset(p,q) = 0$if $p \sim_{MvN} q$ and p, q are projections and $\emptyset(p,q) = 1$ otherwise, and that being an infinite projection is axiomatizable, since p is an infinite projection if and only if $\emptyset(p) = 0$ if and only if $\psi(p) < 1/4$, where

$$\psi(x) = \|x - x^*\| + \|x - x^2\| + \inf_y (\|yy^* - x\| + \|y^*yx - y^*y\| + (1 - \|y^*y - x\|))$$

where y ranges over the set of partial isometries. Since we have that to any projection we can associate the density of its range (both in C_α and B_α), and that we have that $p \leq_{MvN} q$ if and only if the density of p is less or equal than the range of q. Since every possible value for the density is of the form $\aleph_\beta$, for $\beta < a$, the theorem is proved.

We consider abelian C^*-algebras, and particularly the theories of real rank zero abelian C^*-algebras. We give a full classification of the complete theories of abelian real rank zero C^*-algebras in terms of the (discrete first-order) theories of Boolean algebras (recall that a theory is complete if whenever $M \models T$ and $N \models T$ then $M \equiv N$). As an immediate consequence of this classification we find that there are exactly $\aleph_0$ distinct complete theories of abelian real rank zero C^*-algebras. We also give a concrete description of two of these complete theories.

We return to studying saturation. We show how saturation of abelian C^*-algebras is related to the classical notion of saturation for Boolean algebras. We begin by recalling some well-known definitions and properties.

A topological space X such that every collection of disjoint nonempty open subsets of X is countable is said to carry the countable chain condition.

Note that for a compact space being totally disconnected is the same as being 0-dimensional, and this corresponds to the fact that $C(X)$has real rank zero. Moreover any compact Rickart space is 0-dimensional and sub-Stonean, while the converse is false (take for example $\beta\mathbb{N} \setminus \mathbb{N}$). The space X carries the countable chain condition if and only if $C(X)$has few orthogonal positive elements (see Definition (5.3.20)).

Moreover, if $f: X \to Y$ is a continuous map of compact 0-dimensional spaces we have that $\emptyset_f: CL(Y) - CL(X)$ defined as $\emptyset_f(C) = f^{-1}[\mathrm{C}]$is a homomorphism of Boolean algebras. Conversely, for any homomorphism of Boolean algebras $\emptyset: CL(Y) - CL(X)$we can define a continuous map $f_\emptyset: X \to Y$. If f is injective, $\emptyset_f$ is surjective. If f is onto $\emptyset_f$is $1 - \text{to} - 1$ and the same relations hold for $\emptyset$ and $f_\emptyset$.

We recall some basic definitions and facts about Boolean algebras.

Definition(5.3.34)[331]: Let k be an uncountable cardinal. A Boolean algebra B is said to be K-saturated if every finitely satisfiable type of cardinality $< k$ in the first-order language of Boolean algebras is satisfiable.

For atomless Boolean algebras this model-theoretic saturation can be equivalently rephrased in terms of increasing and decreasing chains:

Theorem(5.3.35)[331]: (See [353].) Let B be an atomless Boolean algebra, and k an uncountable cardinal. Then B is K-saturated if and only if for every directed $Y < Z$ such that $|Y| + |Z| < k$ there is $c \in B$ such that $Y < c < Z$.

For the ultracopower construction see [334]. The only use we will make of this tool is the following lemma.

Lemma(5.3.36)[331]: (See [349] and [334].) Let X be a compact Hausdorff space, and let $\mathcal{U}$ be an ultrafilter. Then $C(X)^{\mathcal{U}} \cong C(\sum_{\mathcal{U}} X)$ and $CL(X)^{\mathcal{U}} \cong \mathrm{CL}(\sum_{\mathcal{U}} X)$.

Theorem(5.3.37)[331]: Let A and B be abelian, unital, real rank zero C^*-algebras. Write $A = C(X)$ and $B = C(\mathrm{Y})$,where X and Y are 0-dimensional compact Hausdorff spaces. Then $A \equiv B$ as metric structures if and only if $CL(X) \equiv \mathrm{CL}(\mathrm{Y})$ as Boolean algebras.

Proof: Suppose that $A \equiv B$. By the Keisler-Shelah Theorem (5.3.7) there is an ultrafilter $\mathcal{U}$ such that $A^{\mathcal{U}} \cong B^{\mathcal{U}}$. By Lemma(5.3.36) $A^{\mathcal{U}} \cong C(\sum_{\mathcal{U}} X)$. Thus we have $C(X_{\mathcal{U}}) \cong C(Y_{\mathcal{U}})$, and hence by Gelfand-Naimark $X_{\mathcal{U}}$ is homeomorphic to $Y_{\mathcal{U}}$. Then $CL(\sum_{\mathcal{U}} X) \cong CL(\sum_{\mathcal{U}} Y)$. Applying Lemma(5.3.36) again, we have $CL(\sum_{\mathcal{U}} X) = \mathrm{CL}(\mathrm{X})^{\mathcal{U}}$, so we obtain $CL(X)^{\mathcal{U}} \cong CL(y)^{\mathcal{U}}$, and in particular, $CL(X) \equiv CL(Y)$. The converse direction is similar, starting from the Keisler-Shelah theorem for first-order logic (see [36]).

It is interesting to note that the above result fails when $C(X)$ is considered only as a ring in first-order discrete logic (see [333]).

Corollary(5.3.38)[331]: There are exactly $\aleph_0$ distinct complete theories of abelian, unital, real rank zero C^*-algebras.

Proof: There are exactly $\aleph_0$ distinct complete theories of Boolean algebras; see [340] for a description of these theories.

Corollary(5.3.39)[331]: If X and Y are infinite, compact, 0-dimensional spaces both with the same finite number of isolated points or both having a dense set of isolated points, then $C(X) \equiv C(Y)$.

In particular, let α be any infinite ordinal. Then $C(\alpha + 1) \equiv C(\beta\omega)$. Moreover, if α is a countable limit, $C(2^{\omega}) \equiv C(\beta\omega \setminus \omega) \equiv C(\beta\alpha \setminus \alpha)$.

Proof: Given X, Y as in the hypothesis, again by theorem [340], we have that $CL(X) \equiv CL(Y)$.

In fact, both implications fail. For one direction, recall that $\overline{\mathbb{C}\Gamma} = C(\tilde{\Gamma})$ where $\tilde{\Gamma}$ is the dual group of G. If p is prime then the dual of $\oplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$ is $p^{\mathbb{N}}$, hence

$$\overline{\mathbb{C}^{\oplus}_{\mathbb{N}}\mathbb{Z}/p\mathbb{Z}} \cong \overline{\mathbb{C}^{\oplus}_{\mathbb{N}}\mathbb{Z}/q\mathbb{Z}} = C(2^{\mathbb{N}})$$

for all primes p and q; clearly the groups are not elementary equivalent.

For the forward implication, we given an example pointed out to us by Tomasz Kania. It is known that any two torsion-free divisible abelian groups are elementarily equivalent (see [340]), so in particular, $\mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q}$. The dual group of $\mathbb{Q}$ with the discrete topology is a 1-dimensional indecomposable continuum (see [351]), but the dual group of $\mathbb{Q} \oplus \mathbb{Q}$ is 2-dimensional. Hence $\mathbb{C}\mathbb{Q} \not\equiv \overline{\mathbb{C}(\mathbb{Q} \oplus \mathbb{Q})}$.

We dedicated to the analysis of the relations between topology and countable saturation of abelian C^*-algebras. In particular, we want to study which kind of topological properties the compact Hausdorff space X has to carry in order to have some degree of saturation of the metric structure $C(X)$ and, conversely, to establish properties that are incompatible with the weakest degree of saturation of the corresponding algebra. From now on X will denote an infinite compact Hausdorff space (note that if X is finite then $C(X)_{\leq 1}$ is compact, and so $C(X)$ is fully saturated).

The first limiting condition for the weakest degree of saturation is given by the following lemma:

Lemma(5.3.40)[331]: Let X be an infinite compact Hausdorff space, and suppose that X satisfies one of the following conditions:

(i) X has the countable chain condition;

(ii) X is separable;

(iii) X is metrizable;

(iv) X is homeomorphic to a product of two infinite compact Hausdorff spaces;

(v) X is not sub-Stonean;
(vi) X is Rickart.

Then $C(X)$ is not countably degree-1 saturated.

Proof: First, note that (iii) ⇒ (ii) ⇒ (i). The fact that (i) implies that $C(X)$is not countably degree-1 saturated is an instance of Lemma(5.3.41). Failure of countable degree-1 saturation for spaces satisfying (iv) follows from Theorem(5.3.14), while for those satisfying (v) it follows from [34] and [114]. It remains to consider (vi).

Let X be Rickart. The Rickart condition can be rephrased as saying that any bounded increasing monotone sequence of self-adjoint functions in $C(X)$has a least upper bound in $C(X)$(see [348]).

Consider a sequence $(a_n)_{n\in\mathbb{N}} \subseteq C(X)_1^+$ of positive pairwise orthogonal elements, and let $b_n = \sum_{i<n} a_i$. Then $(b_n)_{n\in\mathbb{N}}$ is a bounded increasing sequence of positive operators, so it has a least upper bound b. Since $\|b_n\| = 1$ for all n, we also have $\|b\| = 1$. The type consisting of $P_{-3}(x) = x$, with $K_{-3} = \{1\}, P_{-2}(x) = b - x$ with $K_{-2} = [1,2], P_{-1}(x) = b - x - 1$with $K^{-1} = \{1\}$ and $P_n(x) = x - b_n - 1$with $K_n = [0,1]$ is consistent with partial solution b_{n+1} (for $\{P_{-3}, \dots, P_n\}$). This type cannot have a positive solution y, since in that case we would have that $y - b_n \geq 0$for all $n \in$ N, yet $b - y > 0$, a contradiction to X being Rickart.

Note that the preceding proof shows that the existence of a particular increasing bounded sequence that is not norm-convergent but does have a least upper bound (a condition much weaker than being Rickart) is sufficient to prove that $C(X)$does not have countable degree-1 saturation. Moreover, the latter argument does not use that the ambient algebra is abelian.

We will compare the saturation of $C(X)$(in the sense of Definition (5.3.8)) with the saturation of $CL(X)$, in the sense of the above theorem.

We are going to obtain the following:

Theorem(5.3.41)[331]: Let X be a compact 0-dimensional Hausdorff space. Then $C(X)$ is countably saturated ⇒ $CL(X)$ is countably saturated and
$CL(X)$ is countably saturated ⇒ $C(X)$ is countably $q.f.$ saturated.

Theorem(5.3.42)[331]: Let X be a compact 0-dimensional Hausdorff space, and assume further that X has a finite number of isolated points. If $C(X)$ is countably degree-1 saturated, then $CL(X)$ is countably saturated. Moreover, if X has no isolated points, then countable degree-*1* saturation and countable saturation coincide for $C(X)$.

Countable saturation of $C(X)$(for all formulas in the language of C^*-algebras) implies saturation of the Boolean algebra, since being a projection is a weakly-stable relation, so every formula in $CL(X)$can be rephrased in a formula in $C(X)$; to do so, write sup for ∀, inf for ∃, $\|x - y\|$for $x \neq y$, and so forth, restricting quantification to projections. This establishes the first implication in Theorem(5.3.41). The second implication will require more effort. To start, we will need the following proposition, relating elements of $C(X)$ to certain collections of clopen sets:

Proposition(5.3.43)[331]: Let$X$ be a compact 0-dimensional space and $f \in C(X)_{\leq 1}$. Then there exists a countable collection of clopen sets $\tilde{Y}_f = \{Y_{n,f}: n \in \mathbb{N}\}$ which completely determines f, in the sense that for each $x \in X$, the value $f(x)$ is completely determined by $\{n: x \in Y_{n,f}\}$.

Proof: Let $\mathbb{C}_{m,1}=\left\{\frac{j_i+\sqrt{-1}j_2}{m}: j_1,j_2\in\mathbb{Z}\wedge\left\|j_1+\sqrt{-}j_2\right\|\leq m\right\}$.

For every $y\in\mathbb{C}_{m,1}$ consider $X_{y,f}=f^{-1}\left(\mathrm{B}_{\frac{1}{\mathrm{m}}}(\mathrm{y})\right)$. We have that each $X_{y,f}$ is a σ-compact open subset of X, so is a countable union of clopen sets $X_{y,f,1},\ldots X_{y,f,n},\ldots,\in$ CL(X) Note that $\bigcup_{y\in\mathbb{C}_{m,1}}\bigcup_{n\in\mathbb{N}}X_{y,f,n}=X$. Let $\tilde{X}_{m,f}=\{X_{y,f,n}\}_{(y,n)\in\mathbb{C}_{m,1}\times\mathbb{N}}\subseteq CL(X)$.

We claim that $\tilde{X}_f=\bigcup_m\tilde{X}_{m,f}$ describes f completely. Fix $x\in X$. For every $m\in\mathbb{N}$ we can find a (not necessarily unique) pair $(y,n)\in\mathbb{C}_{m,1}$ such that $x\in X_{y,f,n}$. Note that, for any $m,n_1,n_2\in\mathbb{N}$ and $y\neq z$, we have that $X_{y,f,n1}\cap X_{z,f,\mathrm{n}_2}\neq\emptyset$ implies $|y-z|\leq\sqrt{2}/m$. In particular, for every $m\in\mathrm{N}$ and $x\in X$ we have

$$2\leq|\{y\in\mathbb{C}_{m,1}:\exists n(x\in X_{y,f,n})\}|\leq 4.$$

Let $A_{x,m}=\{\mathrm{y}\in\mathbb{C}_{m,1}:\exists n(x\in X_{y,f,n})\}$ and choose $a_{x,m}\in A_{x,m}$ to have minimal absolute value. Then $f(\mathrm{x})=\lim_{\mathrm{m}}a_{x,m}$ so the collection $\tilde{X}_f$ completely describes f in the desired sense.

The above proposition will be the key technical ingredient in proving the second implication in Theorem(5.3.41). We will proceed by first obtaining the desired result under the Continuum Hypothesis, and then showing how to eliminate the set-theoretic assumption.

Lemma(5.3.44)[331]: Assume the Continuum Hypothesis. Let B be a countably saturated Boolean algebra of cardinality $2^{\aleph_0}=\aleph_1$. Then $C(S(\mathrm{B}))$ is countably saturated.

Proof: Let $B'\preccurlyeq B$ be countable, and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. By the uniqueness of countably saturated models of size $\aleph_1$, and the Continuum Hypothesis, we have $B'^{\mathcal{U}}\cong\mathrm{B}$. By Lemma(5.3.36) we therefore have $C(S(B)\cong C\left(S(B')\right)^{\mathcal{U}}$, and hence $C(S(\mathrm{B}))$ is countably saturated.

Theorem(5.3.45)[331]: Assume the Continuum Hypothesis. Let X be a compact Hausdorff 0-dimensional space. If $CL(X)$ is countably saturated as a Boolean algebra, then $C(X)$ is quantifier-free saturated.

Proof: Let $\|P_n\|=r_n$ be a collection of conditions, where each P_n is a 2-degree $*$-polynomial in $x_0,\ldots,x_n$, such that there is a collection $F=\{f_{n,i}\}_{n\leq i}\subseteq C(X)_{\leq 1}$, with the property that for all i we have $\left\|P_n(f_0,i,\ldots,f_{n,i})\right\|\in(rn)_{1/i}$ for all $n\leq i$.

For any n, we have that P_n has finitely many coefficients. Consider G the set of all coefficients of every P_n and L the set of all possible 2-degree $*$-polynomials in F U G. Note that for any $n\leq i$ we have that $P_n(f_{0,i},\ldots,f_{n,i})$ G L and that L is countable. For any element $f\in L$ consider a countable collection $\tilde{X}_f$ of clopen sets describing f, as in Proposition(5.3.43).

Since $CL(X)$ is countably saturated and $2^{\aleph_0}=\aleph_1$ we can find a countably saturated Boolean algebra $B\subseteq CL(X)$such that $\emptyset,X\in B$, for all $f\in L$ we have $\tilde{X}_f\subseteq$ B, and $|B|=\aleph_1$.

Let $\iota: B-CL(X)$ be the inclusion map. Then ι is an injective Boolean algebra homomorphism and hence admits a dual continuous surjection $g_\iota: X\longrightarrow S(\mathrm{B})$.

Claim (5.3.46)[331]: For every $f \in L$ we have that $\bigcup \tilde{X}_f = S(B)$.

Proof: Recall that

$$\bigcup \tilde{X}_f = X.$$

By compactness of X, there is a finite $C_f \subseteq \tilde{X}_f$ such that $\bigcup C_f = X$. In particular every ultrafilter on B (i.e., a point of S(B)) corresponds via g_ι to an ultrafilter on $CL(X)$ (i.e., a point of X), and it has to contain an element of C_f. So$\bigcup \tilde{X}_f = S(B)$.

From g_ι as above, we can define the injective map $\phi: C(S(B) \to C(X)$ defined as $\phi(f)(x) = f\left(g_\iota^{-1}(x)\right)$. Note that ϕ is norm preserving: Since ϕ is a unital $*$-homomorphism of C^*-algebra we have that $\|\phi(f)\| \leq \|f\|$. For the converse, suppose that $x \in$ S(B) is such that $|f\ (x)| = r$, and by surjectivity take $y \in X$ such that $g_\iota(y) = x$. Then

$$|\phi(f\ )(y)| = |f\ (g_\iota(g_\iota^{-1}(x)))| = |f\ (x)|$$

For every $f \in L$ consider the function f' defined by $\tilde{X}_f$ and construct the corresponding $*$-polynomials P_n'

Claim (5.3.47)[331]:

(i) $f = \phi(f')\ for\ all\ f \in L.$

(ii) $p_n'(f_{0,i}', \dots, f_{n,i}')i \in (r_n)_{1/i} for\ all\ i\ and\ n \leq i.$

Proof: Note that, since $f_{n,i} = L$ and every coefficient of P_n is in L, we have that $P_n\left(f_{0,i}, \dots, f_{n,i}\right) \in$ L. It follows that condition 1, combined with the fact that ϕ is norm preserving, implies condition 2.

Recall that $g = g_\iota$ is defined by Stone duality, and is a continuous surjective map $g: X \to Y$. In particular g is a quotient map. Moreover by definition, since $X_{q,f,n} \in CL(Y) = B \subseteq CL(X)$, we have that if $x \in Y$ is such that $x \in X_{q,f,n}$ for some $(g, f, n) \in \mathbb{Q} \times L \times \mathbb{N}$, then for all z such that $g(z) = x$ we have $z \in X_{q,f,n}$. Take f and $x \in X$ such that $f(x) \neq \phi(f')(x)$. Consider m such that $|f(x) - \phi(f')(x)| > 2/m$. Pick $y \in \mathbb{C}_{\mathrm{m},1}$ such that there is k for which $x \in \mathrm{X}_{\mathrm{y,f},k}$ and find $z \in Y$ such that $g(z) = x$. Then $z \in X_{y,f,k}$, that implies $f'(z) \in B_{1/m}(y)$ and so $\phi(f')(x) = f'(z) \in B_{1/m}(y)$ contradicting $|f\ (x) - \phi(f')(x)| \geq 2/m$.

Consider now $\{\|P_n'(x_o, \dots, x_n)\| = r_n\}$. This type is consistent type in C(S(B)) by condition 2, and C(S(B)) is countably saturated by Lemma(5.3.44), so there is a total solution $\bar{g}$. Then $h_j = \phi(g_\mathrm{j})$ will be such that $\left\|P_n(\bar{h})\ \right\| = r_n$, since ϕ is norm preserving, proving quantifier-free saturation for $C(X)$.

To remove the Continuum Hypothesis from Theorem (5.3.45) we will show that the result is preserved by σ-closed forcing. We first prove a more general absoluteness result about truth values of formulas, see [106]. For more examples of absoluteness of model-theoretic notions, see [332].

Our result will be phrased in terms of truth values of formulas of infinitary logic for metric structures. Such a logic, in addition to the formula construction rules of the finitary logic we have been considering, also allows the construction of $\sup_n \phi_n$ and $\inf_n \phi_n$ as formulas when the ϕ_n are formulas with a total of finitely many free variables. Two such infinitary logics have been considered. The first, introduced by Ben Yaacov and Iovino in [336], allows the infinitary operations only when the functions defined by the formulas ϕ_n all have a common modulus of uniform continuity; this ensures that the resulting infinitary formula is again uniformly continuous. The second, introduced in [343], does not impose any continuity restriction on the formulas ϕ_n when forming countable infima or suprema; as a consequence, the infinitary formulas of this logic may define discontinuous functions. The following result is valid in both of these logics; the only complication is that we must allow metric structures to be based on incomplete metric space, since a complete metric space may become incomplete after forcing.

Lemma(5.3.48)[331]: Let M be a metric structure, $\phi(\bar{x})$ be a formula of infinitary logic for metric structures, and $\bar{a}$ be a tuple from M of the appropriate length. Let $\mathbb{P}$ be any notion of forcing. Then the value $\emptyset^M(\bar{a})$ is the same whether computed in V or in the forcing extension $V\,[G]$.

Proof: The proof is by induction on the complexity of formulas; the key point is that we consider the structure M in $V[G]$as the same set as it is in V. The base case of the induction is the atomic formulas, which are of the form $P(x)$for some distinguished predicate P. In this case since the structure M is the same in V and in $V[\,G]$, the value of $P^M(\bar{a})$is independent of whether it is computed in V or $V[G]$.

The next case is to handle the case where ϕ is $f(\psi_1, \dots, \psi_n)$, where eachψ_i is a formula and $f\colon [0,1]^n \to [0,1]$ is continuous. Since the formula ϕ is in V, so is the function f. By induction hypothesis each $\psi_i^M(\bar{a})$ can be computed either in V or $V[G]$, and so the same is true of $\emptyset^M\,(\bar{a}) = f\;\psi_1^M(\bar{\alpha}), \dots, f\psi_n^M(\bar{\alpha}))$. A similar argument applies to the case when $\emptyset$ is $\sup_n \psi_n$ or in $f_n\psi_n$.

Finally, we consider the case where $0(\bar{x}) =\ \inf_y \psi(\bar{x}, y)$ (the case with sup instead of inf is similar). Here we have that for every $b \in M, \psi^M(\bar{a}, b)$is independent of whether computed in V or $V[G]$ by induction. In both V and $V[G]$the infimum ranges over the same set M, and hence $\psi^M\,(\bar{a})$is also the same whether computed in V or $V[G]$.

We now use this absoluteness result to prove absoluteness of countable saturation under a-closed forcing.

Proposition(5.3.49)[331]: Let $\mathbb{P}$ be $a\,\sigma$-closed notion of forcing. Let M be a metric structure, and let Φ be a set of (finitary) formulas. Then M is countably Φ-saturated in V if and only if M is countably Φ-saturated in the forcing extension $V[G]$.

Proof: First, observe that since $\mathbb{P}$ is σ-closed, forcing with $\mathbb{P}$ does not introduce any new countable set. In particular, the set of types which must be realized for M to be countably Φ-saturated are the same in V and in $V\,[G]$.

Let $\mathrm{t}(\bar{x})$be a set of instances of formulas from $\Phi$with parameters from a countable set $A \subseteq M$. Add new constants to the language for each $a \in A$, so that we may view t as a type without parameters. Define

$$\phi(\bar{x}) = \inf\{\psi(\bar{x})\colon \psi G\ t\}.$$

Note that $\emptyset^M(\bar{a}) = 0$ if and only if $\bar{a}$ satisfies t in M. This $\emptyset$ is a formula in the infinitary logic of [343]. By Lemma(5.3.48) for any $\bar{a}$ from M we have that $\emptyset^M(\bar{a}) = 0$ in V if and only if $\emptyset^M(\bar{a}) = 0$ in $V[G]$. As the same finite tuples $\bar{a}$ from M exist in V and in $V[G]$, this completes the proof.

Finally, we return to the proof of Theorem(5.3.41). All that remains is to show:

Lemma(5.3.50)[331]: The Continuum Hypothesis can be removed from the hypothesis of Theorem(5.3.45).

Proof: Let X be a 0-dimensional compact space such that $CL(X)$ is countably saturated, and suppose that the

Continuum Hypothesis fails. Let $\mathbb{P}$ be a σ-closed forcing which collapses $2^{\aleph_0}$ to $\aleph_1$ (see [106]). Let $A = C(X)$ and $B = CL(X)$. Observe that since $\mathbb{P}$ is σ-closed we have that A remains a complete metric space in $V[G]$, and by Lemma(5.3.48) A still satisfies the axioms for commutative unital C^*-algebras of real rank zero. Also by Lemma(5.3.48) we have that B remains a Boolean algebra, and the set of projections in A in both V and $V[G]$is B. We note that it may not be true in $V[G]$that $X = S(B)$, or even that X is compact (see [341]), but this causes no problems because it follows from the above that $A = C(S(B))$ in $V[G]$. By Proposition(5.3.49) B remains countably saturated in $V[G]$. Since $V[G]$ satisfies the Continuum Hypothesis we can apply Theorem(5.3.45) to conclude that A is countably quantifier-free saturated in $V[G]$, and hence also in V by Proposition(5.3.49).

With the Continuum Hypothesis removed from Theorem(5.3.45), we have completed the proof of Theorem(5.3.41). It would be desirable to improve this result to say that if $CL(X)$ is countably saturated then $C(X)$ is countably saturated. We note that if the map ϕ in Theorem(5.3.45) could be taken to be an elementary map then the same proof would give the improved conclusion.

We now turn to the proof of Theorem(5.3.42). We start from the easy direction:

Proposition(5.3.51)[331]: If X is a 0-dimensional compact space with finitely many isolated points such that $C(X)$ is countably degree-*1* saturated, then the Boolean algebra $CL(X)$ is countably saturated.

Proof: Assume first that X has no isolated points. In this case we get that $CL(X)$is atomless, so it is enough to see that$CL(X)$satisfies the equivalent condition of Theorem (5.3.35).

Let $Y < Z$ be directed such that$|Y| + |Z| < \aleph_1$. Assume for the moment that both Y and Z are infinite. Passing to a cofinal increasing sequence in Z and a cofinal decreasing sequence in Y, we can suppose that $Z = \{U_n\}_{n\in\mathbb{N}}$ and $Y = \{V_n\}_{n\in\mathbb{N}}$, where

$$U_1 \subsetneq \ldots \subsetneq U_n \subsetneq U_{n+i} \subsetneq \ldots \subsetneq V_{n+i} \subsetneq V_n \subsetneq \ldots \subsetneq V_1.$$

If $\bigcup_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} V_n$ then$\bigcup_{n\in\mathbb{N}} U_n$ is a clopen set, so by the remark following the proof of Lemma(5.3.40), we have a contradiction to the countable degree-1 saturation of

$C(X)$.

For each $n \in \mathbb{N}$, let $p_n = x_{u_n}$ and $q_n = x_{v_n}$, where χ_A denotes the characteristic function of the set A. Then

$$p_1 < \cdots < p_n < p_{n+1} < \cdots < q_{n+1} < q_n < \cdots < q_1$$

and by countable degree-1 saturation there is a positive r such that $p_n < r < q_n$ for every n. In particular $A = \{x \in X: r(x) = 0\}$ and $C = \{x \in X: r(x) = 1\}$ are two disjoint closed sets such that $\overline{\bigcup_{n\in\mathbb{N}} U_n} \subseteq C$ and $\overline{X \setminus \bigcap_{n\in\mathbb{N}} V_n} \subseteq$ A. We want to find a clopen set D such that $A \subseteq D \subseteq X\setminus C$. For each $x \in$ A pick W_x a clopen neighborhood contained in $X\setminus$C.

Then $A \subseteq \bigcup_{x \leq A} W_x$. By compactness we can cover A with finitely many of these sets, say $A \subseteq \bigcup_{i \leq n} W_{x_i} \subseteq X \backslash C$, so $D = \bigcup_{i \leq n} W_{x_i}$ is the desired clopen set.
Essentially the same argument works when either Y or Z is finite. We need only change some of the inequalities from $<$ with $\leq$, noting that a finite directed set has always a maximum and a minimum.
If X has a finite number of isolated points, write $X = Y \cup Z$, where Y has no isolated points and Z is finite. Then $C(X) = C(Y) \oplus C(Z)$ and $\mathrm{CL}(X) = CL(Y) \oplus CL(Z)$. The above proof shows that $CL(Y)$ is countably saturated, and $CL(Z)$ is saturated because it is finite, so $CL(X)$ is again saturated.

To finish the proof of Theorem(5.3.42) it is enough to show that when X has no isolated points the theory of X admits elimination of quantifiers. By Corollary(5.3.39) we have that $C(X) \equiv C(\beta\mathbb{N} \backslash \mathbb{N})$ for such X, so it suffices to show that the theory of $C(\beta\mathbb{N}\backslash\mathbb{N})$ eliminates quantifiers.

Definition(5.3.52)[331]: Let $a_1, \ldots, a_n \in C(X)$(more generally, one can consider commuting operators on some Hilbert space H). We say that $\bar{a} = (a_1, \ldots, a_n)$ is non-singular if the polynomial $\sum_{i=1}^{n} a_i x_i = I$ has a solution $x_1, \ldots, x_n$ in $C(X)$. We define the joint spectrum of $a_1, \ldots, a_n$ to be

$$j\sigma(\bar{a}) = \{\bar{\lambda} \in \mathbb{C}^n : (\lambda_1 - a_1, \ldots, \lambda_n - a_n) \text{ is singular}\}$$

Proposition(5.3.53)[331]: Fix $a_1, \ldots, a_n \in C(X)$. Then $\bar{\lambda} \in j\sigma(\bar{a})$ if and only if $\sum_{i \leq n} |\lambda_i - a_i|$ is not invertible.

Proof: We have that $\bar{\lambda} \in j\sigma(\bar{a})$ if and only if there is $x \in X$ such that $a_i(x) = \lambda_i$ for all $i \leq n$. In particular, $\bar{\lambda} \in j\sigma(\bar{a})$ if and only if $0 \in \sigma(\sum |\lambda_i - a_i|)$ if and only if there is x such that $\sum_{i \leq n} |\lambda_i - a_i|(x) = 0$. Since each $|\lambda_i - a_i|$ is positive we have that this is possible if and only if there is x such that for all $i \leq n, |\lambda_i - a_i|(x) = 0$.

Proposition(5.3.54)[331]: The joint spectrum of an abelian C^*-algebra A is quantifier-free definable.

Proof: First of all recall that, when $\bar{a} = a$, then $j\sigma(\bar{a}) = \sigma(a)$, hence the two definitions coincide for elements. We want to define a quantifier-free definable function $F: A \times \mathbb{C} \to [0,1]$ such that $F(a, \lambda) = 0$ if and only if $\lambda \in \sigma(a)$. Since we showed that $\bar{\lambda} \in \sigma(\bar{a})$ if and only if $0 \in \sigma(\sum_{i \leq n} |\lambda_i - a_i|)$, so, in light of this, we can define a function

$$F_n: A^n \times \mathbb{C}^n \to [0,1]$$

as $F_n(\bar{a}, \bar{\lambda}) = F(|\lambda_i - \alpha_i|, 0)$, hence we have that $F_n(\bar{a}, \bar{\lambda}) = 0$ if and only if $\bar{\lambda} \in j\sigma(\bar{a})$, that implies that the joint spectrum of $\bar{a} \in A^n$ is quantifier-free definable.
To define $\sigma(a)$, recall that, for $f \in A$, the absolute value of f is quantifier-free definable as $|f| = \sqrt{ff^*}$, and for a self-adjoint $f \in A$, its positive part is quantifier-free definable as the function $f_+ = \max(0, f)$. The $F(a, \lambda) = |1 - \|(1 - |a - \lambda .1|)_+\||$ is the function we were seeking.

Theorem(5.3.55)[331]: The theory of $C(\beta\mathbb{N}\backslash\mathbb{N})$ has quantifier elimination. Consequently the theory of real rank zero abelian C^*-algebras without minimal projections has quantifier elimination.

Proof: It is enough to prove that for any $n \in \mathbb{N}$and $\bar{a}, \bar{b} \in C(\beta\mathbb{N}\backslash\mathbb{N})^n$ that have the same quantifier-free type over $\emptyset$ there is an automorphism of $C(\beta\mathbb{N}\backslash\mathbb{N})$ sending a_i to b_i, for all $i \leq n$.

Since $\bar{a}$ and $\bar{b}$ have the same quantifier-free type, we have that $K = j\sigma(\bar{a}) = j\sigma(\bar{b})$. Consider D be a countable dense subset of K and pick $f_{\mathrm{i}}, \dots, f_n, g_1, \dots, g_n \in C(\beta\mathbb{N}\ ) = \ell^\infty(\mathbb{N})$ such that $\forall(d_1, \dots, d_n) \in D$ we have that $F_d = \{m \in \mathbb{N}: \forall i \leq n(f_i(m) = d_i)\}$ and $G_d = \{m \in \mathbb{N}: \forall i \leq n(g_i(m) = d_i)\}$ are infinite, $\pi(f_i) = a_i, \pi(g_i) = b_i$ and, for $m \in \mathbb{N}$ we have that $(f_i(m), \dots, f_n(m)), (g_1(m), \dots, g_n(m)) \in D$.

In particular we have that $\mathbb{N} = \bigcup_{d\in D} F_d = \bigcup_{d \in D} G_d$ and that for all $d \neq d'$ we have $F_d \cap F_{d'} = \emptyset = G_d \cap G_{d'}$, then there is a permutation σ on $\mathbb{N}$ (that induces an automorphism of $C(\beta\mathbb{N} \setminus \mathbb{N})$) such that $f_i \text{ o } \sigma = g_i$for all $i \leq$ n.

The proof of Theorem(5.3.42) is now complete by combining Theorem(5.3.41), Proposition (5.3.51), and Theorem(5.3.55).

Corollary(5.3.56)[370]: Let H be a separable Hilbert space, and let Q_r be the canonical quotient map onto the Calkin algebra. Let $(e_n^r)_{n\in\mathbb{N}}$ be an orthonormal basis for $H,$ and let $S_r \in$ B(H) be the unilateral shift in $\mathfrak{B}(H)$ defined by $S_r(e_n^r) = e_{n+1}^r$ for all n. Then neither S_r nor $Q_r(S_r)$ has a square root, but $1 \otimes S_r \in \mathcal{R}_r \bar{\otimes} B(H)$ does have a square root.

Proof: Suppose that $Q_r(T_r) \in C(H)$ is such that $Q_r(T_r)^2 = Q_r(S_r)$. Since $Q_r(S_r)$ is invertible in the Calkin algebra so is $Q_r(T_r)$. The Fredholm index of S is -1, so if $n \in \mathbb{Z}$ is the Fredholm index of T_r then $2n = -1$, which is impossible. Therefore $Q_r(S_r)$has no square root, and hence neither does S_r.

For the second assertion recall that $R_r \cong M_2(R_r)$, and so

$$\mathcal{R}_r \bar{\otimes} \mathfrak{B}(H) \cong M_2(\mathcal{R}_r \bar{\otimes} \mathfrak{B}(H) = \mathcal{R}_r \bar{\otimes}\ (M_2 \otimes \mathfrak{B}(H)).$$

We view $\mathfrak{B}(H)$as embedded in $M_2 \otimes \mathfrak{B}(\mathrm{H}) = \mathfrak{B}(\mathrm{H}')$for another Hilbert space H$'$; find $(f_n^r)_{n\in\mathbb{N}}$ such that $\{\, e_n^r, f_n^r : n \in \mathbb{N}\, \}$ is an orthonormal basis for H$'$. Let $S_r' \in \mathfrak{B}(\mathrm{H}')$be defined such that $S_r'(e_n^r) = f_n^r$ and $S_r'(f_n^r) = e_{n+1}^r$ for all n. Then $T_r = 1 \otimes S_r' \in R_r \bar{\otimes} S_r(\mathrm{H}')$, and $T_{\mathrm{r}}^2 = 1 \otimes S_r$.

Corollary(5.3.57)[370]: Let A and $A + \epsilon$ be abelian, unital, real rank zero C^*-algebras. Write $A = C(X)$ and $A + \epsilon = C(X + \epsilon)$,where X and $X + \epsilon$ are 0-dimensional compact Hausdorff spaces. Then $A \equiv A + \epsilon$ as metric structures if and only if $CL(X) \equiv \mathrm{CL}(X + \epsilon)$ as Boolean algebras.

Proof: Suppose that $A \equiv A + \epsilon$. By the Keisler-Shelah theorem Theorem (5.3.7) there is an ultrafilter $\mathcal{U}^2$ such that $A^{\mathcal{U}^2} \cong A^{\mathcal{U}^2} + \epsilon$. By Lemma(5.3.36) $A^{\mathcal{U}^2} \cong C(\sum_{\mathcal{U}^2} X)$. Thus we have $C(X_{\mathcal{U}^2}) \cong C(X_{\mathcal{U}^2} + \epsilon)$, and hence by Gelfand-Naimark $X_{\mathcal{U}^2}$ is homeomorphic to $X_{\mathcal{U}^2} + \epsilon$. Then $CL(\sum_{\mathcal{U}^2} X) \cong CL(\sum_{\mathcal{U}^2} X + \epsilon)$. Applying Lemma(5.3.36) again, we have $CL(\sum_{\mathcal{U}^2} X) = \mathrm{CL(X)}^{\mathcal{U}^2}$, so we obtain $CL(X)^{\mathcal{U}^2} \cong CL(X + \epsilon)^{\mathcal{U}^2}$, and in particular, $CL(X) \equiv CL(X + \epsilon)$. The converse direction is similar, starting from the Keisler-Shelah theorem for first-order logic (see [36]).

Corollary(5.3.58)[370]: Let X be a compact 0-dimensional space and $\sum_r f^r \in C(X)_{\leq 1}$. Then there exists a countable collection of clopen sets $\tilde{Y}_{\sum_r f^r} = \{Y_{n,\sum_r f^r}: n \in \mathbb{N}\}$ which completely determines $\sum_r f^r$, in the sense that for each $x^2 \in X$, the value $\sum_r f^r\,(x^2)$ is completely determined by $\{n: x^2 \in Y_{n,\sum_r f^r}\}$.

Proof: Let $\mathbb{C}_{m,1} = \left\{\frac{j_i+\sqrt{-1}j_2}{m}: j_1, j_2 \in \mathbb{Z} \wedge \left\|j_1 + \sqrt{-}j_2\right\| \leq m\right\}$.

For every $y^2 \in \mathbb{C}_{m,1}$ consider $\sum_r X_{y^2,f^r} = \sum_r f^{-r}\left(\mathrm{B}_{\frac{1}{m}}(\mathrm{y})\right)$. We have that each $X_{y^2,\sum_r f^r}$ is a σ -compact open subset of X, so is a countable union of clopen sets $X_{y^2,\sum_r f^r,1}, \ldots X_{y^2,\sum_r f^r,n}, \ldots, \in$ CL(X) Note that $\bigcup_{y^2 \in \mathbb{C}_{m,1}} \bigcup_{n\in\mathbb{N}} X_{y^2,\sum_r f^r,n} = X$. Let $\tilde{X}_{m,\sum_r f^r} = \{X_{y^2,\sum_r f^r,n}\}_{(y^2,n)\in\mathbb{C}_{m,1}\times\mathbb{N}} \subseteq CL(X)$.

We claim that $\tilde{X}_{\sum_r f^r} = \bigcup_m \tilde{X}_{m,\sum_r f^r}$ describes $\sum_r f^r$ completely. Fix $x^2 \in X$. For every $m \in \mathbb{N}$ we can find a (not necessarily unique) pair $(y^2, n) \in \mathbb{C}_{m,1}$ such that $x^2 \in X_{y^2,\sum_r f^r,n}$. Note that, for any $m, n_1, n_2 \in \mathbb{N}$ and $y^2 \neq z^2$, we have that $X_{y^2,\sum_r f^r,n_1} \cap X_{z^2,\sum_r f^r,\mathrm{n}_2} \neq \emptyset$ implies $|y^2 - z^2| \leq \sqrt{2}/m$. In particular, for every $m \in$ N and $x^2 \in X$ we have

$$2 \leq |\{y^2 \in \mathbb{C}_{m,1} : \exists n(x^2 \in X_{y^2,\sum_r f^r,n})\}| \leq 4.$$

Let $A_{x^2,m} = \{\mathrm{y} \in \mathbb{C}_{m,1} : \exists n(x^2 \in X_{y^2,\sum_r f^r,n})\}$ and choose $a_{x^2,m} \in A_{x^2,m}$ to have minimal absolute value. Then $\sum_r f^r$ (x) $= \lim_{\mathrm{m}} a_{x^2,m}$ so the collection $\tilde{X}_{\sum_r f^r}$ completely describes $\sum_r f^r$ in the desired sense.

Corollary(5.3.59)[370]: Let $\mathbb{P}$ be a σ-closed notion of forcing. Let M be a metric structure, and let $\sum_j \Phi_j$ be a set of (finitary) formulas. Then M is countably Φ_j-saturated in V if and only if M is countably $\sum_j \Phi_j$-saturated in the forcing extension $V[G]$.

Proof: First, observe that since $\mathbb{P}$ is σ-closed, forcing with $\mathbb{P}$ does not introduce any new countable set. In particular, the set of types which must be realized for M to be countably $\sum_j \Phi_j$-saturated are the same in V and in $V\,[G]$.

Let $\mathrm{t}(\bar{x})$be a set of instances of formulas from $\sum_j \Phi_j$ with parameters from a countable set $A \subseteq M$. Add new constants to the language for each $a \in A$, so that we may view t as a type without parameters. Define

$$\sum_j \phi_j(\bar{x}) = \sum_j \inf\{\psi_j(\bar{x}): \psi_j G\ t\}.$$

Note that $\sum_j \emptyset_j^M \emptyset_j^M(\bar{a}) = 0$ if and only if $\bar{a}$ satisfies t in M. This $\sum_j \emptyset_j$ is a formula in the infinitary logic of [343]. By Lemma(5.3.48) for any $\bar{a}$ from M we have that $\sum_j \emptyset_j^M(\bar{a}) = 0$ in V if and only if $\sum_j \emptyset_j^M(\bar{a}) = 0$ in $V[G]$. As the same finite tuples $\bar{a}$ from M exist in V and in $V[G]$, this completes the proof.

Corollary(5.3.60)[370]: Fix $a_1^r, \ldots, a_n^r \in C(X)$. Then $\bar{\lambda}^r \in j\sigma(\overline{a^r})$ if and only if $\sum_{i\leq n}|\lambda_i^r - a_i^r||$ is not invertible.

Proof: We have that $\bar{\lambda}^r \in j\sigma(\overline{a^r})$if and only if there is $x \in X$ such that $a_i^r(x) = \lambda_i^r$ for all $i \leq$ n . In particular, $\bar{\lambda}^r \in j\sigma(\overline{a^r})$if and only if $0 \in \sigma(\sum\sum_r |\lambda_i^r - a_i^r|)$if and only if there is x such that$\sum_{i\leq n}\sum_r |\lambda_i^r - a_i^r|(x) = 0$. Since each $\sum_r |\lambda_i^r - a_i^r|$ is positive we have that this is possible if and only if there is x such that for all $i \leq n, \sum_r |\lambda_i^r - a_i^r|(x) = 0$.

Corollary(5.3.61)[370]: The joint spectrum of an abelian C^*-algebra A^r is quantifier-free definable.

Proof: First of all recall that, when $\overline{a^r} = a^r$, then $j\sigma(\overline{a^r}) = \sigma(a^r)$, hence the two definitions coincide for elements. We want to define a quantifier-free definable function $F_r: A^r \times \mathbb{C} \to [0,1]$ such that $F_r(a^r, \lambda_r) = 0$ if and only if $\lambda_r \in \sigma(a^r)$. Since we showed that $\overline{\lambda_r} \in \sigma\,(\overline{a^r})$if and only if $0 \in \sigma(\sum_{i\leq n}\left|\lambda_{r,i} - a_i^r\right|)$, so, in light of this, we can define a function

$$F_{r,n}: A^{rn} \times \mathbb{C}^{\mathrm{n}} \to [0,1]$$

as $F_{r,n}(\overline{a^r}, \overline{\lambda_r}) = F_r(\left|\lambda_{r,i} - \alpha_{\mathrm{i}}\right|, 0)$, hence we have that $F_{r,n}(\overline{a^r}, \overline{\lambda_r}) = 0$ if and only if $\overline{\lambda_r} \in j\sigma(\overline{a^r})$, that implies that the joint spectrum of $\overline{a^r} \in A^{rn}$ is quantifier-free definable.
To define $\sigma(a^r)$, recall that, for $f_r \in A^r$, the absolute value of f_r is quantifier-free definable as $|f_r| = \sqrt{f_r f_r^*}$, and for a self-adjoint $f_r \in A^r$, its positive part is quantifier-free definable as the function $(f_r)_+ = \max(0, f_r)$. The $F_r(a^r, \lambda_r) = |\, 1 - \|(1 - |a^r - \lambda_r.1|)_+\||$ is the function we were seeking.

Chapter 6
Rohlin Property and Borel Complexity

For the Jiang–Su algebra we show the uniqueness up to outer conjugacy of the automorphism with this Rohlin property. We prove that if A is either (i) a separable C*-algebra which is stable under tensoring with $\mathcal{Z}$ or $\mathcal{K}$, or (ii) a separable II_1 factor which is McDuff or a free product of II_1 factors, then the approximately inner automorphisms of A are not classifiable by countable structures.

Section (6.1): Automorphisms of the Jiang–Su Algebra

In the classification program established by Elliott, the Jiang–Su algebra Z is one of the most important C^*-algebras, see [145], and which has been investigated by many people [230], [231], [173], [247], [206]. Toms and Winter proved that all approximately divisible C^*-algebras absorb the Jiang–Sualgebra tensorially, i.e., $\text{A} \cong \text{A} \otimes \text{Z}$ [206]. Rørdam showed that the Cuntz semigroup of a Z- absorbing C^*-algebra is almost unperforated [247]. Recently, Winter has shown some criteria for the absorption of the Jiang–Su algebra [253]. For abstract characterizations of the Jiang–Su algebra in a streamlined way, we refer to the recent by Dadarlat, Rørdam, Toms, and Winter [231], [248], [252]. H. Lin has shown the classification theorem for a large class of C^*-algebras consisting of limits of generalized dimension drop algebras when they absorb the Jiang–Su algebra tensorially [241].

In the case of von Neumann algebras Connes defined the Rohlin property for automorphisms, using a partition of unities consisting of projections, and classified automorphisms of the injective type II_1 factor up to outer conjugacy [229]. Kishimoto gave a method to prove the Rohlin property for automorphisms of AF-algebras for classifying automorphisms up to outer conjugacy, based on Elliott's classification program [232], [235], [239], [240]. For Kirchberg algebras, Nakamura completely classified automorphisms with the Rohlin property by their KK-classes up to outer conjugacy [244]. Recently, Matui has classified automorphisms of AH-algebras with real rank zero and slow dimension growth up to outer conjugacy [243]. For finite actions, Izumi defined the Rohlin property and has shown the classification theory [236], [237]. Recently, Izumi, Katsura, and Matui showed classification results for $\mathbb{Z}^2$-actions with the Rohlin property [177], [238], [242].
The aim is to introduce a kind of the Rohlin property for automorphisms of projectionless C^*-algebras and to give the two main theorems as follows.

Definition (6.1.1)[227]: Let A be a unital C^*-algebra which has a unique tracial state t and absorbs the Jiang–Su algebra Z tensorially, and a be an automorphism of A. We say that a has the weak Rohlin property, if for any $k \in \mathbb{N}$ there exist positive elements $f_2 \in A_+^1, n \in \mathbb{N}$ such that$(f_n)_n \in A_\infty$,

$$\left(a^j(f_n)\right)_n \cdot (f_n)_n = 0, j = 1,2,\dots,k-1$$

$$\tau\left(1 - \sum_{j=0}^{k-1} a^j(f_n)\right) \to 0$$

Here, we denote by $A^\infty$the quotient $\ell^\infty(\mathbb{N}, A) \,/\, c_0(A)$, and $A_{\,\infty}$the central sequence algebra $A^\infty \cap A'$.

We extend a technical condition called property $(SI)$to C^*-algebras which do not necessarily have projections in Definition(6.1.6) Roughly speaking, property (SI) means that if two central sequence of positive elements are given such that one of them is infinitesimally small compared to the other in the sequence algebra, then in fact so in the central sequence algebra.

For a separable, nuclear C^*- algebra A absorbing the Jiang–Su algebra, Rørdam proved that A is purely infinite if and only if A is traceless in [247], and Nakamura proved that the aperiodicity for automorphisms of purely infinite C^*-algebras coincides with the Rohlin property in [244].

If A is a projectionless C^*-algebra with a unique tracial state constructed in [250], the weak Rohlin property is equivalent to the aperiodicity of the automorphism in the GNS-representation associated with the tracial state. A similar definition for finite actions, which is called projection free tracial Rohlin property, has been defined in [1,22]. The first main theorem is an adaptation of the result showed by Hirshberg and Winter in [234] to projectionless C^*-algebras. The second main theorem is an adaptation of the result for UHF algebras showed by Kishimoto in [239] to the Jiang–Su algebra. The proofs of Theorem (6.1.12) and Theorem (6.1.21) will appear mainly in Lemma (6.1.13) and Corollary (6.1.20) We take the two-sided shift automorphism s on the infinite tensor product $\otimes_{n\in\mathbb{Z}}Z \cong Z$ of the Jiang–Su algebra. In Proposition (6.1.14) and Example (6.1.8), we will prove that s has the weak Rohlin property and Z has the property (SI). So, as an application of Theorem (6.1.12), we obtain that:

Corollary (6.1.2)[227]:

$$\left(\underset{n\in\mathbb{N}}{\otimes} z\right) \times_{\sigma} = \mathbb{Z} \otimes Z \cong \left(\underset{n\in\mathbb{N}}{\otimes} z\right) \times_{\sigma} \mathbb{Z}.$$

We recall the generators of the prime dimension drop algebras discovered by Rørdam and Winter in [248]. For projectionless cases we extend the technical property, which was called property (SI) in [249], to projectionless C^*-algebras. By this property, we can obtain the generators defined. We prove Theorem (6.1.12). Using the weak Rohlin property we show the stability for the automorphisms of the Jiang–Su algebra.

When A is a C^*-algebra, we denote by A_{sa} the set of self-adjoint elements of, A, A^1 the unit ball of A, A_+ the positive cone of $A, U(A)$ the unitary group of $A, P(A)$the set of projections of $A, T(A)$the tracial state space of A.

We define an inner automorphism of A by $Adu(a) = uau^*$for $u \in U(A)$ and $a \in A$. We denote by M_{n} the C^* -algebra of $n \times n$ matrices with complex entries and $e_{i,j}^{(n)}$ the canonical matrix units of M_{n}, and we set $e_i^{(n)} = e_{i,i}^{(n)}$. We denote by (m, n) the greatest common divisor of m and $n \in \mathbb{N}$.

The following argument was given by Rørdam and Winter in [247] and [248]. We would like to begin with some definitions about the generators of prime dimension drop algebras and show Proposition (6.1.4). We denote by $I(k, k+1), k \in \mathbb{N}$ the prime dimension drop algebra

$$\{f \in C([0,1]) \otimes M_k \otimes M_{k+1};\ f(0) \in M_k 1_{k+1}, f(1) \in 1_k \otimes M_{k+1}\}$$

and set the self-adjoint unitary

$$u_1 = \sum_{i,j} e_{j,i}^{(k)} \otimes e_{j,i}^{(k)} \in U(M_k \otimes M_k)$$

Define non-unital *-homomorphisms $\rho_0 \colon M_k \otimes M_k \hookrightarrow M_k \otimes M_{k+1}$ by $\rho_0\left(e_{i,j}^{(k)} \otimes e_{l,m}^{(k)}\right) = e_{i,j}^{(k)} \otimes e_{l,m}^{(k)}$, and $\rho :\ C([0,1]) \otimes M_k \otimes M_k \hookrightarrow C([0,1]) \otimes M_k \otimes M_{k+1} by$ by $\rho(f)(t) =$

$\rho_0(f\ (t\ )), t\ \in [0,1]$ .Let $u \in U(C([0,1]) \otimes M_k \otimes M_k)$ be such that $u(0) = 1$ and $u(1) = u_1$and set.

$$v = \sum_{j=1}^{k} e_{1,j}^{(k)} \otimes e_{j,k+1}^{(k+1)}$$
$$w(t) = \rho(u)(t) \otimes cos^{1/2}(\pi t/2) 1_k \otimes e_{k+1}^{(k+1)},$$
$$c_j(t\ ) = w(t) \left(e_{1,j}^{(k)} \otimes 1_{k+1}\right) w^*(t), j = 1,2, \dots, k$$
$$s(t) = sin(\pi t/2) w(t) v, t \in [0,1].$$

Since $c_j(0) = e_{1,j}^{(k)} \otimes 1_{k+1}, c_j(1) = 1_k \otimes e_{1,j}^{(\ k+1)}, s(0) = 0$, and $s(1) = 1_k \otimes e_{1,k+1}^{(k+1)}$, it follows that $c_j, s \in I(k, k+1)$. And we have that

$$w^* w(t) = ww^*(t\ ) = 1_k \otimes \left(1_{k+\ 1} - e_{k+1}^{(k+1)}\right) \oplus cos(\pi t\ /2) 1_k \otimes e_{k+1}^{(k+1)},$$
$$c_i c_j^* = ww^* w \left(e_1^{(k)} e_{j,1}^{(k)} \otimes 1_{k+\ 1}\right) w^* = \delta_{i,j} c_1^2,$$
$$\sum_{j=1}^{k} c_j^* c_i = (w^* w)^2,$$
$$s^* s(t) sin^2(\pi t\ /2) 1_k \otimes e_{k+1}^{(k+1)},$$
$$c_1 s(t) = sin(\pi t\ /\ 2) w \left(e_1^{(k)} \otimes 1_{k+\ 1}\right) w^* w(t) \sum_{i=1}^{k} e_{1,i}^{(k)} \otimes e_{i,k+1}^{(k+1)} = s(t)$$

From these computations, it follows that $\{c_j\}_{j=1}^{k} \cup \{s\}$satiafies

$$c_1 \geqq 0, c_i c_j^* = \delta_{i,j} c_1^2,$$
$$\sum_{j=1}^{k} c_j^* c_i + s^* s = 1, c_1 s = s.$$

To be convenient, we denote by $\mathcal{R}_k$ the above relations on generators of a unital C^*-algebra.Fix a separable infinite-dimensional Hilbert space $\mathcal{H}$, and set

$$\Lambda = \left\{\{c_j'\}_{j=1}^{k} \cup \{s'\} \subset B(\mathcal{H})^1; \text{satisfies} \mathcal{R}_{\mathrm{k}}\right\} \subset 2^{B(\mathcal{H})^1}.$$

For $\lambda \in \Lambda$, let $\mathrm{c}_{\mathrm{j},\lambda} \in \lambda, \lambda, \mathrm{j} = 1, 2, \dots, \mathrm{k}$ and $s_\lambda \in \lambda$ be generators corresponding to $c_j, j = 1, 2, \dots, k$, and s on the relations $\mathcal{R}_{\mathcal{K}}$, and define $\tilde{c}_j = \bigoplus_{\lambda \in \Lambda} c_{j,\lambda}, \tilde{s} = \bigoplus_{\lambda \in \Lambda} S_\lambda \in B(\bigoplus_{\lambda \in \Lambda} \mathcal{H})$.The set $\{\tilde{c}_j\}_{j=1}^{k} \cup \{\tilde{s}\}$ satisfies the relations $\mathcal{R}_{\mathcal{K}}$. Let $C^*(\{\tilde{c}_j\}_{j=1}^{k} \cup \{\tilde{s}\}\ )$ be the C^*-subalgebraof $B(\bigoplus_{\lambda \in \Lambda} \mathcal{H})$generated by $\{\tilde{c}_j\} \cup \{\tilde{s}\}$. Then, we can identify $C^*(\{\tilde{c}_j\}_{j=1}^{k} \cup \{\tilde{s}\}\ )$ with the universal C^*-algebra on a set of generators satisfying the relations $\mathcal{R}_{\mathcal{K}}$.

Proposition(6.1.3)[227]: (See Proposition 5.1 in [248].) The universal C^*-algebra C^*-algebra is isomorphic to $I(k, k+\ 1)$ with $\tilde{c}_j \mapsto c_j$and$\tilde{s} \mapsto s$

Proof: First we show that $C^*\left(\{c_j\}_j \cup \{s\}\right) = I(k, k+\ 1)$ since

$$\sum_{j=1}^{k} c_j^* s s^* c_j + s^* s(t) = sin(\pi t\ /\ 2)(1_k \otimes 1_{k+1}),$$

and $1_k \otimes 1_{k+1} \in C^*\left(\{c_j\}_j \cup \{s\}\right)$ we have that $C([0,1]) \otimes 1_k \otimes 1_{k+1} \subset C^*\left(\{c_j\}_j \cup \{s\}\right)$.Bya partition of unity argument on $[0,1]$, it suffices to show that $C^*\left(\{c_j\}_j \cup \{s\}\right)(i) \cong M_{k+i}, i = 0,1$,and $C^*\left(\{c_j\}_j \cup \{s\}\right)(t) \cong M_k \otimes M_{k+1}, t \in (0,1)$. Since $c_j(0) = e_{1,j}^{(k)} \otimes 1_{k+1}, c_j(1) = 1_k \otimes e_{1,j}^{(k)}, j = 1,2,\ldots,k,$ and $s(1) = 1_k \otimes e_{1,k+1}^{(k+1)}$, it follows that $C^*\left(\{c_j\}_j \cup \{s\}\right)(i) \cong M_{k+i} i = 0,1.$ since

$sc_j s^*(t) = sin^2(\pi t/2)\, cos(\pi t/2) \rho(u)\left(e_{1,1}^{(k)} \otimes e_{1,j}^{(k+1)}\right)\rho(u)^*, j = 1,2,\ldots,k,$

$sc_j s^* c_i(t) = sin^2(\pi t/2)\, cos(\pi t/2) \rho(u)\left(e_{1,i}^{(k)} \otimes e_{1,j}^{(k+1)}\right)\rho(u)^*(t) i, j = 1,2,\ldots,k,$

$$sc_j(t) = sin(\pi t/2)\, cos(\pi t/2) \rho(u)(t)\left(e_{1,j}^{(k)} \otimes e_{1,k+1}^{(k+1)}\right) = 1,2,\ldots,k,$$

for $t \in (0,1)$,we have that $C^*(\{c_j\}_j \cup \{s\})(t) = M_k \otimes M_{k+1}$ for $t \in (0,1)$.

Set $A = C^*(\{\tilde{c}_j\}_{j=1}^k \cup \{\tilde{s}\})$.

Let $\Phi : A \to I(k, k+1)$ be the *-homomorphism defined by $\Phi(\tilde{c}_j) = c_j$ and $\Phi(\tilde{s}) = s$. It remains to show that Φ is injective. Let $(\pi, \mathcal{H})$ be an irreducible representation of A. Because for any $a \in A$ there exists an irreducible representation of A which preserves the norm of a (see [83]), it suffices to show that there exists a representation φ of $I(k,k+1)$ on $\mathcal{H}$ such that $\varphi(c_j) = \pi(\tilde{c}_j)$ and $\varphi(s) = \pi(\tilde{s})$.

Set

$$\tilde{b} = \sum_{j=1}^{k} \tilde{c}_j^* \tilde{s}\, \tilde{s}^* \tilde{c}_j + \tilde{s}^* \tilde{s}.$$

By the following computations, we see that $\tilde{b}$ is in the center of A. Since $\{\tilde{c}_j\}_j \cup \{\tilde{s}\}$ satisfies therelations $\mathcal{R}_k$, in particular $\tilde{c}_1^2 = \tilde{c}_j \tilde{c}_j^*$,we have that

$$\tilde{b}\tilde{c}_j = \tilde{s}\, \tilde{s}^* \tilde{c}_j + \tilde{s}^* \tilde{s} \tilde{c}_j,$$
$$\tilde{c}_j \tilde{b} = \tilde{s}\tilde{s}^* \tilde{c}_j + \tilde{c}_j \tilde{s}^* \tilde{s},$$
$$\tilde{s}^* \tilde{s}\, \tilde{c}_j = \tilde{c}_j - \tilde{c}_1^2 \tilde{c}_j = \tilde{c}_j \tilde{s}^* \tilde{s},$$
$$\tilde{b}\tilde{s} = \tilde{b}\, \tilde{c}_1 \tilde{s} = \tilde{s}\tilde{s}^* \tilde{s} + \tilde{s}\tilde{s}^* \tilde{s},$$
$$\tilde{s}\tilde{b} \sum \tilde{s}\tilde{c}_j^* \tilde{s}\, \tilde{s}^* \tilde{c}_j + \tilde{s}\tilde{s}^* \tilde{s}$$
$$\tilde{s}^* \tilde{s}^2 = \tilde{c}_1 \tilde{s} - \tilde{c}_1^3 \tilde{s} = \tilde{s} - \tilde{s} = 0$$
$$\tilde{s}^* \tilde{s} \tilde{c}_j^* \tilde{s}\tilde{s}^* \tilde{c}_j = \tilde{c}_j^* \tilde{s}\tilde{s}^* \tilde{c}_j - \tilde{c}_j^* \tilde{c}_j \tilde{c}_j^* \tilde{s}\tilde{s}^* \tilde{c}_j = 0.$$

Hence $[\tilde{b}, \tilde{c}_j] = 0$ and $[\tilde{b}, \tilde{s}] = 0$.

Set $\bar{c}_j = \pi(\tilde{c}_j), \tilde{s} = \pi(\tilde{s})$, and $\bar{b} = \pi(\tilde{b})$. Since $0 \leq \tilde{b} \leq 1$, we have that$sp(\tilde{b}) = [0,1]$ and obtain $\beta \in [0,1]$such that $\beta 1 = \bar{b}$. When $\beta = 0$ we have $\bar{s} = 0$. Thus $\{\bar{c}_j\}_j$ satisfies the relations for matrix units $\{e_{1,j}^{(k)}\}$ of M_k, and then $\mathcal{H} = \mathbb{C}^k$. Set $V_0 : I(k, k+1) \to M_k$as the irreducible representationat 0. Since $V_0(c_j) = e_{1,j}^{(k)}$

we obtain a unitary u. and define $\varphi = Adu_0 0 V_0$.

When $\beta = 1$, by the following computations, we see that $\bar{c}_j^* \bar{c}_j, j = 1,2,\ldots,n$, and $\bar{s}^* \bar{s}$ are orthogonal projections. Since $\bar{b} = 1$, we have that $\sum \bar{c}_j^* (1 - \bar{s}\, \bar{s}^*)\, \bar{c}_j = 0$. Then it follows that $\bar{c}_1^2 = \bar{s}\bar{s}^*, \bar{c}_1^4 = \bar{c}_1^2, (\bar{c}_j^*\bar{c}_j)^2 = \bar{c}_j^*\bar{c}_1^2\bar{c}_j = \bar{c}_j^*\bar{c}_j$ and $(\bar{s}^*\bar{s})^2 = \bar{c}_j^*\bar{c}_1^2\bar{s} = \bar{s}^*\bar{s}$.From$\sum \bar{c}_j^*\bar{c}_j + \bar{s}^*\bar{s} = 1$ it follows that $\bar{c}_j^*\bar{c}_j, j = 1,2,\dots,k$ and $\bar{s}^*\bar{s}$ are mutually orthogonal projections. Hence $\{\bar{c}_j\}_j \cup \{\bar{s}\}$

satisfies the relations for matrix units $\{e_{1,j}^{(k+1)}\}_{j=1}^{k+1} of\ M_{k+1}$. Then we see that $\mathcal{H} = \mathbb{C}^{(k+1)}$ andcan define $\varphi : I(k, k+1) \to B(\mathcal{H})$ as the irreducible representation of $I(k, k+1)$ at $t = 1$upto unitary equivalence.

When $0 < \beta < 1$, by the following computations, we see that

$$E_{i,j} = (\beta(1-\beta))^{-1}\bar{c}_i^*\bar{s}\bar{c}_j^*\bar{c}_j\bar{s}^*\bar{c}_i,$$
$$E_{j,k+1} = (\beta(1-\beta))^{-1}\bar{c}_j^*\bar{c}_j\bar{s}^*\bar{s},$$

$i,j = 1,2,\dots,k,$ are mutually orthogonal projections. Since $b\bar{s} = \bar{s}\bar{s}^*\bar{s}$ and $(1-\bar{b})\bar{s}^*\bar{s} = (1-\bar{s}^*\bar{s})\bar{s}^*\bar{s}$, we have that

$$E_{i,j}^2 = (\beta(1-\beta))^{-2} = \bar{c}_i^*\bar{s}\bar{c}_j^*\bar{c}_j\bar{s}^*\bar{c}_j^2\bar{s}\bar{c}_j^*\bar{c}_j\bar{s}^*\bar{c}_i$$
$$= (\beta(1-\beta))^{-2}\bar{c}_i^*\bar{s}\bar{s}^*\bar{s}\bar{c}_j^*\bar{c}_j\bar{c}_j^*\bar{c}_j\bar{s}^*\bar{c}_i$$
$$= \beta^{-1}(1-\beta)^{-2}\bar{c}_i^*\bar{s}(1-\bar{s}^*\bar{s})\bar{c}_j^*\bar{c}_j\bar{s}^*\bar{c}_i = E_{i,j},$$

$$E_{j,k+1}^2 =$$

$$(\beta(1-\beta))^{-2}\left(\bar{c}_j^*\bar{c}_j\right)^2(\bar{s}^*\bar{s})^2$$
$$= \beta^{-1}(1-\beta)^{-1}\bar{c}_j^*\bar{c}_j(1-\bar{s}^*\bar{s})\bar{s}^*\bar{s}E_{j,k+1},$$

$$\sum_{i,j} E_{i,j} + \sum_j E_{j,k+1} = (\beta(1-\beta))^{-1}\left(\sum_{i=1}^{k}\bar{c}_i^*\bar{s}(1-\bar{s}^*\bar{s})\bar{s}^*\bar{c}_i + (1-\bar{s}^*\bar{s})\bar{s}^*\bar{s}\right)$$
$$= \beta^{-1}\left(\sum_{i=1}^{k}\bar{c}_i^*\bar{s}\,\bar{s}^*\bar{c}_i + \bar{s}^*\bar{s}\right) = 1.$$

Set

$$F_{i,j} = \beta^{-1}(1-\beta)^{-1/2}\bar{s}\bar{c}_j\bar{s}^*\bar{c}_i$$
$$F_{j,k+1} = (\beta(1-\beta))^{-1/2}\bar{s}\bar{c}_j, i,j = 1,2,\dots,k.$$

Then it follows that $F_{i,j}^*F_{i,j} = E_{i,j}, F_{j,k+1}^*F_{j,k+1} = E_{j,k+1}, F_{i,j}F_{i,j}^* = E_{1.1}$ and $F_{j,k+1}F_{j,k+1}^* = E_{1.1}$. Thus $\{F_{i,j}\}_{i,j} \cup \{F_{j,k+1}\}_j$ satisfies the same relations as matrix units $\{e_{1,j}^{(k)} \otimes e_{1,j}^{(k+1)}\}_{i,j} \cup \left\{e_{1,j}^{(k)} \otimes e_{1,k+1}^{(k+1)}\right\}_j$ of $M_k \otimes M_{k+1}$. It is not so hard to see that $\beta^{1/2}\sum_{j=1}^{k} F_{j,k}^*F_{j,k+1} = \bar{s}, \beta\sum_{j=1}^{k} F_{1,j}^*F_{i,j} = \overline{s}\bar{s}^*\bar{c}_i$, and $\beta\bar{c}_i = \overline{s}\bar{s}^*\bar{c}_i + \bar{s}^*\overline{s}\bar{c}_i$ .Then we have that $C^*(\{F_{i,j}\} \cup \{F_{j,k+1}\}) = C^*(\{\bar{c}_j\}_j \cup \{\bar{s}\})$ And $\mathcal{H} = \mathbb{C}^{k(k+1)}. Let V_\beta$ be the irreducible representation of $I(k, k+1)$ at $t \in (0,1)$ with $\sin^2(\pi t/2) = \beta$. Then $V_\beta o\Phi(\bar{b}) = \beta$ and there exists a unitary u_β such that

$$F_{i,j} = \beta^{-1}(1-\beta)^{-1/2}Adu_\beta\ oV_\beta\left(sc_j\ s^*c_i\right),$$
$$F_{j,k+1} = (\beta(1-\beta))^{-1/2}Adu_\beta oV_\beta(scj).$$

Hence, we have that $Adu_\beta\ oV_\beta(c_j) = \bar{c}_j$ and $Adu_\beta oV_\beta(s) = \bar{s}$ and obtain $\varphi = Adu_\beta oV_\beta$. This completes the proof.

Definition (6.1.4)[227]: Let A be a unital C^*-algebra and $\tau \in T(A)$. We recall the dimension function d_τ and define $\bar{d}_\tau: A_+^1 \to \mathbb{R}_+$by

$$d_\tau(f) = \lim_{n\to\infty} \tau((1/n+f)^{-1}f),$$
$$\bar{d}_\tau(f) = \lim_{n\to\infty} \tau(f^n), f \in A_+^1.$$

Lemma (6.1.5)[227]: For $f_n \in A_+^1, n \in \mathbb{N}$ with $(f_n)_n \in A_\infty$ and an increasing sequence $m_n \in \mathbb{N}, n \in \mathbb{N}$ with $m_n \nearrow \infty$,it follows that:

(i) If $\lim_{n\to\infty} \max_{\tau\in T}(A)\tau(f_n) = 0$ then there exist $\tilde{f}_n \in A_+^1, n \in \mathbb{N}$ such that $(\tilde{f}_n)_n = (f_n)_n$ And $\lim_{n\to\infty} \max_{\tau\in T}(A) d_\tau(\tilde{f}_n) = 0$.

(ii) There exist $\tilde{f}_n \in A_+^1, n \in \mathbb{N}$ such that $(\tilde{f}_n)_n = (f_n)_n$ and $\lim inf_{n\to\infty} \min_{\tau\in T(A)} \tilde{d}_\tau(\tilde{f}_n) \lim inf_n \min_\tau \tau(f_n^{m_n})$.

(iii) If A absorbs Z tensorially, then there exist $f_n^{(i)} \in A_+^1, i = 0,1, n \in \mathbb{N}$ such that $(f_n^{(i)})_n \in A_\infty, f_n^{(0)} f_n^{(1)} = 0, (f_n^{(i)})_n (f_n)_n, i = 0,1$, and

$$\lim inf_{n\to\infty} \min_{\tau\in T(A)} \bar{d}_\tau (f_n^{(i)}) \geq \lim inf_n \min_\tau \tau(f_n^{m_n})/2.$$

Proof: (i) Let $\varepsilon_n > 0$ be such that $\varepsilon_n \searrow 0$ and $\max_{\tau\in T(A)} \tau(f_n)\varepsilon_n^2$. Set

$$g_\varepsilon(t) = \begin{cases} (1-\varepsilon)^{-1}(t-\varepsilon), \varepsilon \leq t \leq 1, \\ 0, \qquad\qquad 0 \leq t \leq \varepsilon, \end{cases}$$

and $\tilde{f}_n = g_{\varepsilon_n}(f_n)$. Then we have that $\|\tilde{f}_n - f_n\| \leq \varepsilon_n$ and $\varepsilon_n \lim_{m\to\infty} (1/m + \tilde{f}_n)^{-1}\tilde{f}_n \leq f_n$, which implies that $d_\tau(\tilde{f}_n) \leq \varepsilon_n$, for any $\tau \in T(A)$.

(ii) Let $\varepsilon_n > 0$ be such that $\varepsilon_n \searrow 0$, and$(1-\varepsilon_n)^{m_n} \to 0$. Set

$$g_\varepsilon(t) = \begin{cases} (1-\varepsilon)^{-1}t, 0 \leq t \leq 1-\varepsilon, \\ 1, \qquad\qquad 1-\varepsilon \leq t \leq 1, \end{cases}$$

And $\tilde{f}_n = g_{\varepsilon_n}(f_n)$. Then we have that $\|\tilde{f}_n - f_n\| \leq \varepsilon_n$ and $f_n^{m_n} = f_n^{m_n}(lim_{l\to\infty}\tilde{f}_n^l + \chi([0,1-\varepsilon_n))(f_n)) \leq lim_{l\to\infty}\tilde{f}_n^l + (1-\varepsilon_n)^{m_n}$ (where $\chi(S)$ means the characteristic function of S), which implies that $\tau(f_n^{m_n}) d_\tau(\tilde{f}_n) + (1-\varepsilon_n)^{m_n}$, for any $\tau \in T(A)$.

(iii) Set $c = \lim inf_{n\to\infty} \min_{\tau\in T(A)}\tau(f_n^{m_n})$. Since$A \cong A \otimes_{n\in\mathbb{N}} Z$, we obtain $l_n \in \mathbb{N}$ and $\bar{f}_n \in (A \otimes_{j=1}^{l_n} Z)_+^1$such that $l_n \nearrow \infty$ and $m_n\|\bar{f}_n - f_n\| \to 0$, which implies that $(\bar{f}_n)_n \in A_\infty$ and $\lim inf_{n\to\infty} \min_{\tau\in T(A)}\tau(\bar{f}_n^{m_n}) = c$. By an argument as in the proof of (ii), we obtain $\bar{f}_n \in (A \otimes_{l=1}^{l_n} Z)_+^1, n \in \mathbb{N}$ such that $(\bar{f}_n)_n = (\bar{f}_n) = (f_n)$ and $\lim inf_{n\to\infty} \min_{\tau\in T(A)} \bar{d}_\tau (\tilde{f}_n) \geq c$. Let $g_n^{(i)} \in Z_+^1, i = 0,1, n \in \mathbb{N}$ be such that $g_n^{(0)} g_n^{(1)} = 0, \lim inf_n \bar{d}_{\tau Z}(g_n^{(i)}) = 1/2, i = 0,1$, where τ_Z means the unique tracial state of Z. Set $f_n^{(i)} = \tilde{f}_n \otimes g_n^{(i)} \in A \otimes_{l=1}^{l_n+1} Z$. Since $l_n \nearrow \infty$,it follows that $(f_n^{(i)})_n \in A_\infty$, and since $\tau((f_{na}^{(i)})P) = \tau(\tilde{f}_n^P \otimes 1)\tau Z((g_n^{(i)})P), P \in \mathbb{N}, \tau \in T(A)$, it follows that

$$\lim \inf_n \min_\tau \bar{d}_\tau(f_n^{(i)}) = \lim \inf_n \min_\tau \bar{d}_\tau(\tilde{f}_n)\, \bar{d}_{\tau Z}(g_n^{(i)})\, c/2, i = 0,1.$$

In [249],we have defined a technical condition, called property (SI), forC^*-algebra with non-trivial projections. For C^*-algebras which do not necessarily have projections, we generalize this technical condition in the following.

Definition (6.1.6)[227]: We say that A has the property (SI), when for any en and $f_n \in A_+^1, n \in N$satisfying the following conditions:$(e_n)_n, (f_n)_n \in A_\infty$,

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(e_n) = 0,$$
$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(f_n^n) > 0,$$

there exist $s_n \in A^1, n \in \mathbb{N}$, such that$(s_n)_n \in A_\infty$ and

$$s_n^* s_n = (e_n), (f_n s_n) = (s_n).$$

Example (6.1.7)[227]: any UHF algebra has the property (SI).

Proof: Let B be a UHF algebra, and let en and $f_n \in B_+^1, n \in \mathbb{N}$satisfy the conditions in the property (SI). Let $B_n, n \in \mathbb{N}$ be an increasing sequence of matrix subalgebras of B such that $(\bigcup_{n\in\mathbb{N}} B_n) = B$ and $1_{B_n} = 1_B$. For any B_n, we denote by $\Phi_n: B \to B_n' \cap B$ the conditional expectations [249]. By $(e_n)_n, (f_n)_n \in B_\infty$, we obtain a slow increasing sequence $m_n \in \mathbb{N}, n \in \mathbb{N}$ such that $m_n \nearrow \infty, m_n \leq n, (\Phi_{m_n}(e_n))_n = (e_n)_n$, and

$$\lim_{n\to\infty} m_n \|\Phi_{m_n}(f_n) - f_n\| = 0\,,$$

and we obtain a fast increasing sequence $l_n, n \in \mathbb{N}$, and $\bar{e}_n\,, \tilde{f}_n \in (B_{m_n}' \cap B_{l_n})_+^1$ such that $m_n < l_n, (\bar{e}_n)_n = (\Phi_{m_n}(e_n))_n$, and $\lim_{n\to\infty} m_n \| \bar{f}_n - \Phi_{m_n}(f_n)\| = 0$. Then we have that $\lim \tau(\bar{e}_n) = 0$ and $\lim \| f_n^{m_n} - f_n^{m_n} \| = 0$, which implies that $\lim\inf f \tau(\bar{f}_n^{m_n}) = \lim\inf f\ \tau(f_n^{m_n}) > 0$.

By Lemma (6.1.5)(i), we obtain $\tilde{e}_n \in (B_{m_n}' \cap B_{l_n})_+^1$ such that $(\tilde{e}_n)_n = (e_n)_n$ and $\lim d_\tau(\tilde{e}_n) = 0$. By Lemma (6.1.5)(ii), we obtain $\tilde{f}_n \in (B_{m_n}' \cap B_{l_n})_+^1$ such that $(\tilde{f}_n)_n = (f_n)_n$ and

$$\lim_{n\to\infty} \inf f \bar{d}_\tau(\tilde{f}_n) \geq \lim\inf f \tau\big(\bar{f}_n^{m_n}\big) > 0.$$

Taking a large $N \in \mathbb{N}$,we have that

$$d_{T_{r_n}}(\tilde{e}_n) = d_\tau(\tilde{e}_n) < \bar{d}_\tau\ (\tilde{f}_n) = \tilde{d}_{T_{r_n}}\ (\tilde{f}_n), n \geq N,$$

where T_{r_n} is the normalized trace of $B_{m_n}' \cap B_{l_n}$. Then, we obtain $s_n (B_{m_n}' \cap B_{l_n})^1$ such that $s_n^* s_n = \tilde{e}_n, \tilde{f}_n s_n = s_n$, hence we have that $(s_n)_n \in B_\infty, (s_n^* s_n)_n = (e_n)_n$, and$(f_n s_n)_n = (s_n)_n$.

The following proposition is motivated by Lemma (6.1.6) in [243]. Combining this proposition and the above example we conclude Example (6.1.8).

Example (6.1.8)[227]: The Jiang–Su algebra has the property (SI).

In order to prove the above proposition, we define the projectionless C^*-algebra Z_k for $k \in \mathbb{N}\backslash\{1\}$by

$$Z_k\ = f \in C[0,1] \otimes M_{k^\infty} \otimes M_{(k+1)^\infty};$$
$$f(0) \in M_{k^\infty} \otimes M_{(k+1)^\infty}, f(1)\ \in 1_{k^\infty} \otimes M_{(k+1)^\infty}.$$

This projectionless C^*-algebra $Z\ \ k$ was introduced by Rørdam and Winter in [26,30] as a mediator between C^*-algebras absorbing UHF algebras and C^*-algebras absorbing the Jiang–Sualgebra.

Proposition (6.1.9)[227]: Let A be a unital C^*-algebra absorbing the Jiang–Su algebra Z tensorially. If $A \otimes B$ has the property (SI) for any UHF algebra B, then A also has the property (SI).

Proof. Suppose that e_n and $f_n \in A_+^1, n \in \mathbb{N}$ satisfy the conditions in the property (SI). Let k be a natural number with $k \geq 2, B^{(i)}$ the UHF algebra of rank $(k+i)^\infty$, and $\Phi^{(i)}$ the canonical unital embeddings of $A \otimes B^{(i)}$ into $A \otimes B^{(0)} \otimes B^{(1)}, i = 0,1$.

By Lemma (6.1.5) there exist $f_n^{(i)} \in A_+^1, i = 0,1, n \in \mathbb{N}$, such that $(f_n^{(i)})_n \in A_\infty, (f_n^{(0)}\ )_n\ (f_n^{(1)}\ )_n = 0, (f_n^{(i)})_n\ (f_n)_n$, and $lim\ inf_{n\to\infty} min_{\tau\in T(A)\tau}\ (f_n^{(i)^n}) \geq$ $lim\ inf_n min_\tau \bar{d}_\tau(f_n^{(i)}) \geq lim\ inf_n min_\tau\ \ \tau(f_n^n)/2 > 0, i = 0,1.$

Applying the property (SI) of $A \otimes B^{(i)}$ to $e_n \otimes 1_{B^{(i)}}$ and $f_n^{(i)} \otimes 1_{B^{(i)}} \in A \otimes B_+^{(i)^1}$ we obtain $s_n^{(i)} \in A \otimes B(i)^1, i = 0,1, n \in \mathbb{N}$ such that$(s_n^{(i)})_n \in (A \otimes B^{(i)})_\infty$,
$\left(s_n^{(i)^*} s_n^{(i)}\right)_n = (e_n \otimes 1_{B^{(i)}}), \left(f_n^{(i)} \otimes 1_{B^{(i)}} \cdot s_n^{(i)}\right)_n = \left(s_n^{(i)}\right)$.
Note that$(\Phi^{(i)}(s_n^{(i)^*} s_n^{(i)}))_n = (e_n \otimes 1_{B^{(0)} \otimes B^{(1)}})_n, (f_n \otimes 1_{B^{(0)} \otimes B^{(1)}})_n \cdot$

$$(\Phi^{(i)}(s_n^{(i)}))_n = (\Phi^{(i)}(s_n^{(i)}))_n, \text{and} (\Phi^{(0)}(s_n^{(0)}))_n^* (\Phi^{(1)}(s_n^{(1)}))_n = 0.$$

Define $s_n \in A \otimes Z_k^1, n \in \mathbb{N}$by

$$s_n(t) = cos(\pi\, t/2)\Phi^{(0)}\left(s_n^{(0)}\right) + sin(\pi\, t/2)\Phi^{(1)}\left(s_n^{(1)}\right), t \in [0,1].$$

Since

$$(s_n^* s_n(t))_n = \left(cos^2(\pi\, t/2)\Phi^{(0)}\right)\left(s_n^{(0)^*} s_n^{(0)}\right) + sin^2(\pi\, t/2)\Phi^{(1)}\left(s_n^{(1)^*} s_n^{(1)}\right) + cos$$
$$\cdot\, sin\,(\pi\, t/2)\left(\Phi^{(0)} s_n^{(0)}\right)^* \Phi^{(1)}\left(s_n^{(1)}\right) + \Phi^{(1)}\left(s_n^{(1)}\right)^* \Phi^{(0)} s_n^{(0)})))_n$$
$$= (e_n \otimes 1_{B^{(0)} \otimes B^{(1)}})_n, t \in [0,1],$$

$(f_n \otimes 1_{B^{(0)} \otimes B^{(1)}})_n (s_n(t)) = (s_n(t)), t \in [0,1]$, and $Lip(s_n) = \pi, n \in \mathbb{N}$, it follows that

$$(s_n^* s_n)_n = (e_n \otimes 1_{Z_k})_n, (f_n \otimes 1_{Z_k})_n (s_n)_n = (s_n)_n.$$

Set$\iota: A_\infty \hookrightarrow (A \otimes Z_k)_\infty$by $\iota((a_n)_n) = (a_n \otimes 1_{Z_k})_n$. Since $A \cong A \otimes_{n \in \mathbb{N}} Z$ and $Z_k \subset unital\, Z$, for any finite subset $F \subset A_\infty$, we obtain a unital embedding $\Phi_F: (A \otimes Z_k)_\infty \hookrightarrow A_\infty$ such that $\Phi F o \iota(x) = x, x \in F$.
Define $s = \Phi_{\{(e_n),(fn)\}}((s_n)_n) \in A_\infty$, then we conclude that $s^* s = (e_n)$ and$(f_n)_n s = s$.

We show Theorem (6.1.12). We denote by A_α the fixed point algebra of $\alpha \in Aut(A)$ and by α_∞ the automorphism of A_∞ induced by α. In the following Lemma (6.1.10), mimicking in [234], we use the weak Rohlin property to obtain a set of elements in$(A_\infty)\alpha_\infty$ which satisfies the same relations as $\{c_j\}_{j=1}^k\, in\, \mathcal{R}_k$. After that, applying the property (SI) and the weak Rohlin property, we obtain the generators of prime dimension drop algebras in$(A_\infty)\alpha_\infty$.

Lemma (6.1.10)[227]: Let A be a unital separable C^*-algebra which has a unique tracial state τ and absorbs the Jiang–Su algebra tensorially. Suppose that $\alpha \in Aut(A)$ has the weak Rohlin property.
Then for any $k \in \mathbb{N}$ there exist $c_{j,n} \in A, j = 1,2,\ldots,k, n \in \mathbb{N}$ such that
$(c_{j,n})_n \in (A_\infty)_{\alpha_\infty}^1$,
$(c_{1,n}) \geqq 0, (c_{i,n})_n (c_{j,n})^* = \delta_{i,j}\, (c_{1,n})^2$,

$$\left\|(c_{1,n})_n\right\| = 1, \qquad \left\|1 - \sum_{j=1}^{k} (c_{j,n})_n^* (c_{j,n})\right\| = 1, \lim_{n\to\infty} \tau\left(c_{1,n}^n\right) = 1/k$$

which implies $\lim_{n\to\infty} \tau(1_A - \sum_{j=1}^k c_{j,n}^* c_{j,n}) = 0)$.

Proof: Let $\Phi_m, m \in \mathbb{N}$ be the unital embeddings of Z into $A \otimes_{m \in \mathbb{N}} Z \cong A$ defined by

$$\Phi_m(x) = 1_A \otimes 1_{\otimes_{i=1}^m} Z \otimes x \otimes 1_{\otimes_{i=m+2}^\infty} Z, x \in Z,$$

And Φ be the unital embedding of Z into A^∞ defined by $\Phi(x) = (\Phi_m(x))_m, x \in Z$. Note that $\tau(\Phi_m(x)) = \tau Z(x), m \in \mathbb{N}, x \in Z$.

In the definition of $c_j \in I(k,k+1)$, replacing cos with cos^{1/n^2} we obtain $c_n^{(j)} \in I(k,k+1) \subset Z, j = 1,2,\ldots,k$ such that

$$c_n^{(1)} \geqq 0, c_n^{(i)} c_n^{(j)^*} = \delta_{i,j} c_n^{(1)^2},$$

$$\left\| \left(c_n^{(1)}\right)_n \right\| = 1, \left\| 1 - \sum_{j=1}^{k} c_n^{(j)^*} c_n^{(j)} \right\| = 1, \tau Z\left(c_n^{(1)^n}\right) \nearrow 1/k.$$

Let $\varepsilon_n > 0, n \in \mathbb{N}$ be such that $\varepsilon_n \searrow 0$ and $\tau Z\left(c_n^{(1)^n}\right) \nearrow 1/k - \varepsilon_n, n \in \mathbb{N}$.
Let k_n and $l_n \in \mathbb{N}, n \in \mathbb{N}$ be such that $l_n \nearrow \infty$and $l_n^2 < k_n$.Since $\alpha \in Aut(A)$ has the weak Rohlin property and A and Z are separable C^*-algebras, we obtain $f^{(n)} = (f_m^{(n)})_m \in (A \cup \bigcup_{j\in\mathbb{Z}} \alpha_\infty^j(\Phi(Z)))' \cap A^\infty$ such that $\|f^{(n)}\| = 1$,

$$\alpha_\infty^p\left(f^{(n)}\right) f^{(n)} = 0, and\ \tau\left(f^{(n)^n}\right) > 1/(2(k_n + l_n) + 1) - \varepsilon_m,$$

for all p with $0 < |p| \leq 2(k_n + l_n)$ and for all $r \geq m$. Note that any subsequence of $(f_m^{(n)})_m$ satisfies the above conditions. Then, taking a subsequence of $(f_m^{(n)})_m$, we may suppose that

$$\left|\tau\left(\Phi_m\left(c_n^{(1)}\right)^n \cdot f_m^{(n)^n}\right) - \tau Z\left(c_n^{(1)^n}\right)\tau\left(f_m^{(n)^n}\right)\right| < \varepsilon_m.$$

For $p \in \mathbb{Z}$, define $a_{p,n} \geq 0$ by

$$a_{p,n} = \begin{cases} 1 - (|p| - k_n)/\, l_n, k_n < |p| \leq k_n + l_n, \\ 1, \qquad |p| \leq k_n\,, \\ 0, \qquad k_n + l_n < |p|, \end{cases}$$

and define completely positive maps $\varphi_n : Z \to A_\infty$,by
$\varphi_n(x) = \sum_{|p| \leq k_n + l_n} a_{p,n} \alpha_\infty^p(\Phi(x)) \alpha_\infty^p\left(f^{(n)}\right)$.
Then we have that

$$\|\alpha_\infty(\varphi_n(x)) - \varphi_n(x)\| = \left\| \sum_{|p| \leq k_n + l_n} (a_{p,n} - a_{p-1,n}) \cdot \alpha_\infty^p(\Phi(x)) \alpha_\infty^p\left(f^{(n)}\right) \right\| = \|x\|/l_n,$$

$x \in Z,$

$$\varphi_n\left(c_n^{(i)}\right) \varphi_n\left(c_n^{(j)}\right)^* = \sum_{|p| \leq k_n + l_n} a_{p,n}^2 \alpha_\infty^p\left(\Phi\left(c_n^{(i)}\, c_n^{(j)^*}\right)\right) \alpha_\infty^p\left(f^{(n)}\right)^2 = \delta_{i,j} \varphi_n\left(c_n^{(1)}\right)^2,$$

$n \in \mathbb{N},$

$\left\|\varphi_n\left(c_n^{(1)}\right)\right\| = 1$, and $\left\|1 - \sum_{j=1}^{k} \varphi_n\,(c_n^{(j)})^* \varphi_n(c_n^{(j)})\right\| = 1$.

Let $c_{n.m}^{(j)} \in A^1, j = 1,2,\ldots,k, m \in \mathbb{N}$ be component of $\varphi_n\left(c_n^{(j)}\right)\left(i.e., \left(c_{n,m}^{(j)}\right)_m = \varphi_n\left(c_n^{(j)}\right) \in A_\infty\right)$ with $c_{n,m}^{(j)} \geqq 0$ then we have that

$$\begin{aligned} \lim_{m\to\infty} \tau\left(c_{n,m}^{(j)\ n}\right) &= \liminf_m \sum_{|P| \leq k_n + l_n} a_{P,n}^n \tau\left(a^P\left(\Phi_m\left(c_n^{(j)}\right)^n\right) a^P\left(f_m^{(n)}\right)^n\right) \\ &> (2k_n + 1) \liminf_m \tau\left(\Phi_m\left(c_n^{(j)}\right)^n . f_m^{(n)^n}\right) \\ &= (2k_n + 1) \liminf_m \tau z\left(c_n^{(1)}\right)^n \tau\left(f_m^{(n)^n}\right) \end{aligned}$$

$$> \frac{(2k_n+1)}{(2(k_n+l_n)+1)}(1/k-\varepsilon_n) \to_{n\to\infty} 1/k.$$

Let F_n be an increasing sequence of finite subsets of A_1 with $\overline{\bigcup_n F_n} = A^1$. By the above conditions, we obtain an increasing sequence $m_n \in \mathbb{N}, n \in \mathbb{N}$ such that

$$\left\|\left[c_{n,m_n}^{(j)}, x\right]\right\| < \varepsilon,\ \ x \in F_n,$$

$$\left\|\alpha\left(c_{n,m_n}^{(j)}\right) - c_{n,m_n}^{(j)}\right\| < 1/l_n + \varepsilon_n,$$

$$c_{n,m_n}^{(i)} c_{n,m_n}^{(j)\,*} - \delta_{i,j} c_{n,m_n}^{(1)\,2} < \varepsilon_n, \qquad i,j = 1,2,\dots,k,$$

$$\left\|c_{n,m_n}^{(1)}\right\| > 1-\varepsilon_n, \left\|1 - \sum_{j=1}^{k} c_{n,m_n}^{(j)\,*} c_{n,m_n}^{(i)}\right\| > 1-\varepsilon_n,$$

$$\tau\left(c_{n,m_n}^{(1)\,n}\right) > \frac{2k_n+1}{2(k_n+l_n)+1}(1/k-\varepsilon_n).$$

Define $c_{j,n} = c_{n,m_n}^{(j)}, j = 1,2,\dots,k$, then we have that $(cj,n)_n \in (A_\infty)\alpha_\infty$,

$$(c1,n)_n \geqq 0, (c_{i,n})_n(c_{j,n})^* = \delta_{i,j}(c_{1,n})^2,$$

$$\|(c1,n)_n\| = 1, \left\|1 - \sum_{j=1}^{k}(cj,n)_n^*(cj,n)\right\| = 1, \lim_{n\to\infty}\tau\left(c_{1,n}^n\right) = 1/k.$$

By the technique in the proof of Lemma 4.6 in [240] we obtain a generator s in (A∞)α∞ satisfying the relations in $\mathcal{R}_k$ together with$\{(cj,n)_n\}$ above. In the proof of the following proposition, $x \approx_\varepsilon y$ means $\|x-y\| < \varepsilon$.

Proposition(6.1.11)[227]: Let Abe a unital separable C^*-algebra which has a unique tracial state τ, absorbs the Jiang–Su algebra Z tensorially, and has the property(SI). Suppose that $\alpha \in \mathrm{Aut}(A)$ has the weak Rohlin property. Then for any $k \in \mathbb{N}$ there exists a set of norm-one elements$\{cj\}_{j=1}^{k} \cup \{s\}\ in(A_\infty)\alpha_\infty$ satisfying $\mathcal{R}_k$.

Proof. By Lemma (6.1.10) we obtain $c_m^{(j)} \in A^1, j = 1,2,\dots,k, m \in \mathbb{N}$ such that$(c_m^{(j)})_m \in (A_\infty)\alpha_\infty, (c_m^{(1)})_m \geq 0, (c_m^{(i)})_m(c_m^{(j)})^* = \delta_{i,j}(c_m^{(1)})^2, (c_m^{(j)})_m = 1, 1 - \sum_{j=1}^{k}(c_m^{(j)})^*(c_m^{(j)}) = 1$, $\lim_{m\to\infty}\tau(c_m^{(1)^m}) = 1/k$, and $\lim_{m\to\infty}\tau(1-\sum c_m^{(j)^*} c_m^{(j)}) = 0$, where τ is a unique tracial state Of A. Let $\varepsilon_m > 0, m \in \mathbb{N}$ be such that εm0 and $\tau(c_m^{(1)^m})1/k - \varepsilon m$.

Because of the weak Rohlin property of $\alpha \in \mathrm{Aut}(A)$ we obtain $f_m^{(l)} \in A_+^1, l, m \in \mathbb{N}$, such that $(f_m^{(l)})m \in A_\infty$ and

$$\alpha_\infty^p\left(\left(f_m^{(l)}\right)_m\right)\left(f_m^{(l)}\right) = 0, p = 1,2,\dots,l-1,$$

$$\left\|\left[f_r^{(l)}, c_m^{(1)}\right]\right\| < \varepsilon_m, \qquad r \geq m,$$

$$\tau f_r^{(l)^m} > 1/l - \varepsilon_m, r \geq m.$$

Note that any subsequence of $(f_m^{(l)})_m$ satisfies the above conditions. Since τ is the unique tracial state, taking a subsequence of $(f_m^{(l)})_m$ we may suppose that $\tau\left((c_m^{(1)} f_m^{(l)})^m\right) \approx \varepsilon_m \tau(c_m^{(1)^m} f_m^{(l)^m}) \approx_{\varepsilon_m} \tau(c_m^{(1)^m})\tau\,(f_m^{(l)^m})1/(kl) - 2\varepsilon_m$. Set then we have that $(g_m^{(l)})_m \in A_{\infty+}^1, l \in \mathbb{N}$ and $\lim\,\mathrm{in}f_{m\to\infty}\,\tau(g_m^{(l)^m})1/(kl)$.

By the property (SI) of A, we obtain $s_m^{\prime(l)} \in A^1, m \in \mathbb{N}$, such that $(s_m^{\prime(l)})_m \in A_\infty, (s_m^{\prime(l)^*} s_m^{\prime(l)}) = (1 - \sum_{j=1}^k s_m^{(l)^*} c_m^{(j)})$, and$(g_m^{(l)} s_m^{\prime(l)}) = (s_m^{\prime(l)})$. Remark that $(g_m^{(l)}) = (f_m^{(l)})(c_m^{(1)}) \leq (f_m^{(l)}), (g_m^{(l)})(c_m^{(1)}), (f_m^{(l)})(s_m^{\prime(l)}) = (s_m^{\prime(l)})$, and $(c_m^{(1)})_m(s_m^{\prime(l)}) = (s_m^{\prime(l)})$.Let $L_n \in \mathbb{N}$ be such that $2\, L_n^{-1/2} < \varepsilon_n (L_n \nearrow \infty)$ and define

$$s_m^{(L_n)} = L_n^{-1/2} \sum_{p=0}^{L_n-1} \alpha^P \left(s_m^{\prime(L_n)}\right).$$

Then we have that

$$\left\| \alpha s_m^{(L_n)} - s_m^{(L_n)} \right\| \leq 2L_n^{-1/2} < \varepsilon_n, m \in \mathbb{N}.$$

Let $m_n \in \mathbb{N}, n \in \mathbb{N}$ be an increasing sequence with $m_n \nearrow \infty$ such that $(s_{m_n}^{(L_n)})_n \in A_\infty, \left\| s_{m_n}^{(L_n)} \right\| \leq 1 + \varepsilon_n, \left\| f_{m_n}^{(L_n)} s_{m_n}^{\prime(L_n)} - s_{m_n}^{\prime(L_n)} \right\| < \varepsilon_n / L_n, \left\| \alpha^P\ (f_{m_n}^{(L_n)} f_{m_n}^{(L_n)} \right\| < \varepsilon_n / L_n, \left\| s_{m_n}^{\prime(L_n)^*} s_{m_n}^{\prime(L_n)} - \sum_{j=1}^k c_{m_n}^{(j)^*} c_{m_n}^{(j)}) \right\| \leq \varepsilon_n, \left\| \alpha^P (c_{m_n}^{(j)}) - c_{m_n}^{(j)} \right\| < \varepsilon_n / (2k), j = 1,2, \ldots, k, p = 1,2, \ldots, L_n - 1$, and $\left\| c_{m_n}^{(1)} s_{m_n}^{(L_n)} - s_{m_n}^{(L_n)} \right\| < \varepsilon_n / L_n$, and set $s_n = s_{m_n}^{(L_n)}$.
Then we have that $\alpha_\infty((s_n)_n) = (s_n) \in A_\infty$,

$$\begin{aligned} s_n^* s_n &\approx_{2\varepsilon_n} L_n^{-1} \left(\sum_{p=0}^{L_n-1} \alpha^P s_{m_n}^{\prime(L_n)^*} f_{m_n}^{(L_n)} \right) \left(\sum_{q=0}^{L_n-1} \alpha^q f_{m_n}^{(L_n)} s_{m_n}^{\prime(L_n)} \right) \\ &\approx_{\varepsilon_n} L_n^{-1} \sum_{p=0}^{L_n-1} \alpha^P s_{m_n}^{\prime(L_n)^*} f_{m_n}^{(L_n)^2} s_{m_n}^{\prime(L_n)^*} \\ &\approx_{\varepsilon_n} L_n^{-1} \sum_{p=0}^{L_n-1} \alpha^P \left(s_{m_n}^{\prime(L_n)^*} s_{m_n}^{\prime(L_n)} \right) \\ &\approx_{\varepsilon_n} L_n^{-1} \sum \alpha^P \left(1 - \sum_{j=1}^k c_{m_n}^{(j)^*} c_{m_n}^{(j)} \right) \\ &\approx_{\varepsilon_n} 1 - \sum_{j=1}^k c_{m_n}^{(j)^*} c_{m_n}^{(j)}, \end{aligned}$$

$(s_n)_n = 1$ and $(c_{m_n}^{(1)})(s_n) = (s_n)$. Hence we conclude that $\{(c_{m_n}^{(j)})_n\}_{j=1}^k \cup \{(s_n)_n\} \subset (A_\infty)_{\alpha_\infty}^1$ and they satisfy the relations $\mathcal{R}_k$.

Theorem(6.1.12)[227]: Let A be a unital separable C^*-algebra which does not necessarily have projections, has a unique tracial state, and absorbs the Jiang–Su algebra tensorially. Suppose that A has property (SI) and a is an automorphism of A with the weak Rohlin property. Then $A \times_\alpha \mathbb{Z}$ also absorbs the Jiang–Su algebra tensorially.

Proof. Applying Proposition 2.2 in [206] to$(A \rtimes_\alpha \mathbb{Z})^\infty$ it suffices to show the following lemma.

Lemma(6.1.13)[227]: Let A be a unital separable C^*-algebra which does not necessarily have projections, has a unique tracial state, and absorbs the Jiang–Su algebra tensorially.

Suppose that a has the Property (SI) And $\alpha \in \mathrm{Aut}(A)$ has the weak Rohlin property. Then for any $k \in \mathbb{N}$ there exists a unital$*$-homomorphism Φ_k from $I(k,k+1)$to$(A_\infty)\alpha_\infty$.

Proof. By Proposition (6.1.11) we obtain a set of norm-one generators$\{c_j\}_{j=1}^{k} \cup \{s\}$ in $(A_\infty)_{\alpha_\infty}$ satisfying $\mathcal{R}_k$. Then, by Proposition (6.1.3), we conclude the above lemma.

We see the following proposition as an example of automorphisms with the weak Rohlin property, hence we conclude Corollary (6.1.2).

Proposition (6.1.14)[227]: The two-sided shift automorphism σ on $\otimes_{n\in\mathbb{Z}} Z$ has the weak Rohlin property.

Proof: Identify $\otimes_{n\in\mathbb{Z}} Z$ with Z.Let$(\pi,\mathcal{H})$ be the GNS-representation of Z associated with the unique tracial state τZ and $\bar{\alpha}$ the weak extension of $\alpha \in Aut\,(Z)$ on $(Z)''$. Because of Theorem 1.2 in [250], we have shown that the weak Rohlin property is equivalent to the aperiodicity in the GNS-representation associated with the unique tracial state. Then it suffices to show that $\bar{\sigma}^k \neq Ad\,V$ for any $V \in U(\pi(Z))''$ and any $\in \mathbb{N}$. In particular, since $\otimes_{i=1}^{k} Z \cong Z$, it suffices to show that $\bar{\sigma} \neq AdV$ for any $V \in U(\pi(Z)'')$.

Assume that there exists $V_\sigma \in U(\pi(Z)'')$such that $Ad\,V_\sigma = \bar{\sigma}$. Note that $\bar{\sigma}^k(V_\sigma) = V_\sigma$ for any $k \in \mathbb{N}$. However we see that $\pi(Z)''_{\bar{\sigma}} = \mathbb{C}1$, this is acontradiction.

Indeed, for any $V \in U(\pi(Z)'')$ With $\bar{\sigma}(V) = V$ and any $\varepsilon > 0$, we obtain $N \in \mathbb{N}$ and $v \in \otimes_{-N}^{N} Z \subset \otimes_{n\in\mathbb{Z}} Z (= Z)$ such that $\|V - \pi(v)\|_2 < \varepsilon$, where $\|x\|\,2 := \tau Z(x^*x)^{1/2}$, then $\left\|\bar{\sigma}^k\,(V\,) - \pi o \sigma^k(v)\right\| 2 < \varepsilon$ for all $k \in \mathbb{N}$. Hence, for any $a \in Z^1$, it follows that $\|[V,\pi(a)]\|_2 < 2\varepsilon$. Since$\varepsilon$ is arbitrary, we conclude that $V \in \pi(Z) \cap \pi(Z)'' = \mathbb{C}1$.

Using the weak Rohlin property, we show the stability for automorphisms of the Jiang–Su algebra Theorem (6.1.15) and prove Theorem (6.1.21).

First, we recall the generalized determinant introduced by P. de la Harpe and G. Skandalis (see [233], [238], [246]). Let A be a unital C^*-algebra with a unique tracial stateτ. For any piecewise differentiable path $\xi: [0,1] \to U(A)$, we define

$$\tilde{\Delta}_\tau(\xi\,) = \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau\,(\xi(t)\xi^*(t))dt \in \mathbb{R}$$

When $\xi(0) = \xi(1) = 1$wehavethat$\tilde{\Delta}_\tau(\xi\,) \in \tau(K_0(A))$. For any $u \in U_0(A)$, there exists a piecewise differentiable path $\xi_u: [0,1] \to U(A)$ such that $\xi_u(0) = 1, \xi_u(1) = u$. The generalized determinant Δ_τ associated with the tracial state τ is the map from $U_0(A) to \mathbb{R}/\tau\,(K_0(A))$ defined by $\Delta_\tau(u) = \tilde{\Delta}(\xi_u) + \tau(K_0(A))$. Note that Δ_τ is a group homomorphism.

Mimicking the proof of Lemma 6.2 in [238] we prove the following proposition. Hereinafter, we let log be the standard branch defined on the complement of the negative real axis.

Lemma (6.1.15)[227]: Let$A$ be a unital C^*-algebra with a unique tracial state τ.

(i) For $u_1, u_2 \in U(A)$with, $\|u_i - 1\| < 1/2, i = 1,2$ it follows that

$$\tau o log(u_1u_2) = \tau o log(u_1) + \tau o log(u_2).$$

(ii)For u_1, u_2, and $v \in U(A)$with $\|u_1 - u_2\| < 1/2$and $\|v - 1\| < 1/4$, it follows that

$$\tau o \log(u_1 v u_2^* v^*) = \tau o \log(u_1 u_2^*).$$

Proof: (i) Let $h_i \in A_{sa}$ be such that $\exp\,(2\pi\sqrt{-1}h_i) = u_i, i = 1,2,$ and $h_3 \in A_{sa}$ be such that $\exp\,(2\pi\sqrt{-1}h_3) = u_1u_2$. Set $u(t) = \exp(2\pi\sqrt{-1}th_1) \cdot \exp(2\pi\sqrt{-1}th_2), w(t) = \exp(2\pi\,\sqrt{-1}th_3), t \in [0,1]$. Since$\|1 - u(t)\| < 1, \|1 - w(t)\| < 1,$ and$\|1 - w^*u(t)\| <$

$2, t \in [0,1]$, we can define $h \in C([0,1]) \otimes A_{sa}$ by $h(t) = \log(w^*u(t)), t \in [0,1]$, then u and ware homotopic, by $H(s,t) = w(t)\exp((1-s)h(t))$with fixed endpoints $H(s,0) = 1$ and $H(s,1) = w(1)$. Hence, we have that

$$\tau o \log(u_1u_2) = 2\pi\sqrt{-1}\tau(h_3) = \int_0^1 \tau\,(\dot{w}w^*(t))dt = \int_0^1 \tau\,(\dot{u}u^*(t))dt$$
$$= 2\pi\sqrt{-1}\tau(h_1+h_2) = \tau o log(u_1) + \tau o log(u_2).$$

(ii) Set $U_1 = v^*u_1vu_1^*, U_2u_1u_2^*$, then it follows that $\|U_i - 1\| < 1/2, i = 1,2$. Applying (i), since $\tau o\, log(U_1) = \tau o log(v^*) + \tau o log(u_1vu_1^*) = 0$ we have that $\tau o log(U_1U_2) = \tau o log(U_1) + \tau o log(U_2) = \tau o log(U_2)$.

Proposition(6.1.16)[227]: Let B be the UHF algebra of rank k^∞, where $k \in \mathbb{N}\backslash\{1\}, \tau$ the unique tracial state of $B, \beta \in Aut(B)$, and $u_n \in U(B), n \in \mathbb{N}$ with $(u_n)n \in B_\infty$. Suppose that $\beta \in Aut(B)$has the Rohlin property and

$$\Delta_\tau(u_n) = 0, \qquad \text{for any } n \in \mathbb{N}.$$

Then there exist $v_n \in U(B), n \in \mathbb{N}$ such that$(v_n)_n \in B_\infty$,

$$(v_n\beta(v_n)^*)_n = (u_n)_n, \qquad \tau^\circ log(v_n\beta(v_n)^*u_n^*) = 0, for\ any\ n \in \mathbb{N}.$$

The following lemma was essentially proved in [233].

Proof. Because $\beta \in Aut(B)$ has the Rohlin property in [239], there exist $v_n' \in U(B), n \in N$such that $(v_n')_n \in B_\infty$, and

$$(v_n'\beta(v_n')^*)_n = (u_n).$$

By the assumption and

$$\frac{1}{2\pi\sqrt{-1}}\tau o\log(v_n'\beta(v_n')^*u_n^*) + \tau(K_0(B))$$
$$= \Delta_\tau(v_n'\beta(v_n')^*u_n^*) = \Delta_\tau(v_n'\beta(v_n')^*) - \Delta_\tau(u_n) = -\Delta_\tau(u_n),$$

we have that

$$\frac{1}{2\pi\sqrt{-1}}\tau o log(v_n'\beta(v_n')^*u_n^*) + \tau(K_0(B)), n \in \mathbb{N}.$$

Since B is the UHF algebra of rank k^∞, we obtain $l_n \in \mathbb{N}$ and $m_n \in \mathbb{Z}$ such that $(m_n, k) = 1$and

$$k^{-l_n}m_n = -\frac{1}{2\pi\sqrt{-1}}\tau o\log(v_n'\beta(v_n')^*u_n^*) + \tau(K_0(B)).$$

Set $\lambda_n = exp(2\pi\sqrt{-1}k^{-l_n}m_n)$, then we have that $\lambda_n \to 1$, by$(v_n'\beta(v_n')^*u_n^*)_n = 1$. By the Rohlin property of $\beta \in Aut(B)$, there exist $p_n \in P(B)$ and $z_n \in U(B)$, $n \in \mathbb{N}$ such that $(p_n)_n \in B_\infty, (z_n)n = 1_{B_\infty}$, and

$$\sum_{j=0}^{k^{l_n}-1}(\mathrm{Adz_n o\beta})^j(p_n) = 1_B\,.$$

Define

$$\bar{v}_n = \sum_{j=0}^{k^{l_n}-1}\exp\left(2\pi\sqrt{-1}jk^{-l_n}m_n\right)\cdot(\mathrm{Adz_n o\beta})^j(p_n), v_n = v_n'\bar{v}_n \in U(B), n \in \mathbb{N}.$$

Taking a subsequence of$(pn)_n$and$(z_n)_n$, we may suppose that $(\bar{v}_n)_n \in B_\infty$.

Then it follows that $(v_n)_n \in B_\infty$.By the definition of $\bar{v}_n$ we have that $\bar{v}_n Adz_n o\beta(\bar{v}_n)^* = \lambda_n$ and

$$(v_n\beta(v_v)^*u_n^*)_n = (v_n Adz_n o\beta v_n^* u_n^*)_n = (\lambda_n v_n'\beta(v_n')^* u_n^*)_n = 1.$$

And, by Lemma (6.1.15), we have that

$$\tau o log(v_v\beta(v_n)^*u_n^*) = \tau olog(v_n\, Adz_n o\beta(v_n)^*u_n^*)$$
$$= \tau olog(\bar{v}_n Adzno\beta(\bar{v}_n)^* Adz_n o\beta(v_n')^* u_n^* v_n')$$
$$= 2\pi\sqrt{-1}k^{-l_n}m_n + \tau ologv_n'\beta(v_n')^*u_n^* = 0, n \in \mathbb{N}.$$

Lemma (6.1.17)[227]: For any $c > 0$ there exists $c' > 0$ such that the following holds. Let B be a UHF algebra, τ the unique tracial state of B. Suppose that $\tilde{u}_n \in U(C([0,1]) \otimes B), n \in \mathbb{N}$ satisfy that

$$(\tilde{u}_n)_n \in (C([0,1]) \otimes B)_\infty, (\tilde{u}_n(i))_n = 1, i = 0,1,$$
$$\tilde{\Delta}_\tau(\tilde{u}_n) = 0, \qquad \tau olog(\tilde{u}_n(i)) = 0, \qquad i = 0,1, n \in \mathbb{N},$$

and Lip $(\tilde{u}_n) < c, n \in \mathbb{N}$. Then there exist $y_n \in U(C([0,1]^2) \otimes B), n \in \mathbb{N}$ such that $(y_n)_n \in (C([01]^2) \otimes B)_\infty$,

$$y_n(0,t) = 1B, y_n(1,t) = \tilde{u}_n(t), t \in [0,1],$$
$$y_n(s,i) = exp(log\, \tilde{u}_n(i)s), i = 0,1, s \in [0,1],$$

And Lip$(y_n) < c', n \in \mathbb{N}$.

Proof. Set $\partial E = \{(s,t) \in [0,1]^2; \{s,t\} \cap \{0,1\} = \emptyset\}$. By Proposition 4.6 in [238], for $c > 0$, we obtain $c' > 0$ satisfying that: for any AF-algebra A and for any $z \in U(C(\partial E) \otimes A)$ with $z(0,0) = 1, Lip(z) < c$, and $[z]_1 = 0 \in K_1(C(\partial(E)) \otimes A)$, there exists $\tilde{z} \in U(C([0,1]2) \otimes A)$ such that $\tilde{z}|_{\partial E} = z$ and $Lip(\tilde{z}) < c'$. Suppose that $\tilde{u}_n \in U(C([0,1]) \otimes B)$ satisfies the conditions in the lemma. Define $U_n \in U(C(\partial E) \otimes B)$ by

$$Un(s,t) = \begin{cases} 1, & s = 0, \\ \tilde{u}_n(t), & s = 1, \\ exp\big(log\big(\tilde{u}_n(i)\big)s\big), & t = i, i = 0,1. \end{cases}$$

Then we have that Lip $(U_n) < c$ for any $n \in \mathbb{N}$. By the assumption, regarding $U_n \in U(C(T) \otimes B)$,we have that $[U_n]_1 = \tilde{\Delta}_\tau(U_n) = 0\; in\; \tau(K_0(B))$ .Let $B_n, n \in \mathbb{N}$ be an increasing sequence of matrix subalgebras of B with $1_{B_n} = 1_B$ and $\overline{\bigcup B_n} = B$. Since $(U_n)_n \in U\,((C(\partial E) \otimes B)_\infty)$, slightly modifying U_n , we obtain an increasing sequence $m_n \in N, n \in \mathbb{N}$ and $U_n' \in U(C(\partial E) \otimes (B_{m_n}' \cap B))$ such that $m_n \nearrow \infty, (U_n')_n = (U_n)_n, U_n'(0,0) = 1$, and $Lip(U_n') < c$. Since $B_{m_n}' \cap B$ has the unique tracial state $\tau|_{B_{m_n}'} \cap B$,it follows that $[U_n']_{K_1(B_{m_n}' \cap B)} = \tilde{\Delta}_{\tau_{(B_{m_n}' \cap B)}}(U_n') = \tilde{\Delta}_{\tau B}(U_n') = \tilde{\Delta}_{\tau B}(U_n) = 0$,then we obtain $\widetilde{U}_n \in U(C([0,1]^2) \otimes (B_{m_n}' \cap B)), n \in \mathbb{N}$ such that $\widetilde{U}_n|_{\partial E} = U_n'$ and Lip $(\widetilde{U}_n) < c'$. Then we have that $(\widetilde{U}_n)_n \in (C([0,1]^2) \otimes B)_\infty$. Since $\widetilde{U}_n|_{\partial E} = U_n', n \in \mathbb{N}$, slightly modifying $\widetilde{U}_n$ on ∂E ,we obtain $y_n \in U\,(C([0,1]^2) \otimes B)\; n \in \mathbb{N}$ and $\varepsilon > 0$ such that$(y_n)_n = (\widetilde{U}_n)_n, y_n|_{\partial E} = U_n$, and $Lip(y_n) < c' + \varepsilon$ for any $n \in \mathbb{N}$.

As in the proof of Proposition 2.2 in [206], unital$*$-homomorphisms from $I(k, k+1) to (A_{\infty_{\alpha_\infty}}$ obtained in Lemma (6.1.13) implies the following lemma.

Lemma(6.1.18)[227]: Let A be a unital separable C*-algebra which does not necessarily have projections, has a unique tracial state, and absorbs the Jiang–Su algebra tensorially. Suppose that A has the property (SI) and a is an automorphism of A with the weak Rohlin property. Then there exists a unital embedding of Z into $(\mathrm{A}_\infty)_{\alpha_\infty}$.

Theorem (6.1.19)[227]: Suppose that $\alpha \in \mathrm{Aut}(Z)$ has the weak Rohlin property and $u_n \in U(Z), n \in \mathbb{N}$ satisfy $(u_n)_n \in Z_\infty$ and that

$$\Delta_{\tau_Z}(u_n) = 0, \text{for any } n \in \mathbb{N}.$$

Then there exist $v_n \in U(Z), n \in \mathbb{N}$ such that$(v_n)_n \in Z_\infty$ and

$$v_n\alpha(v_v)^*{}_n = (u_n)_n.$$

The following lemma is a direct adaptation of Proposition 4.6 in [238].

Proof. Let $Z_k, B^{(i)}$and $\Phi^{(i)}, i = 0,1,$ be the projectionless C^*-algebra, the UHF algebras, and the unital embeddings in the proof of Proposition (6.1.9).

Because of $Z \otimes B^{(i)} \cong B^{(i)}$, the Røhlin property of $a \otimes id_{B^{(i)}} \in \mathrm{Aut}(Z \otimes B^{(i)}$, and

$$\Delta_{\tau_{z\otimes B^{(i)}}}\left(u_n\otimes 1_{B^{(i)}}\right) = 0, \ \ in\, \mathbb{R}/\tau\left(k_0\left(B^{(i)}\right)\right), n \in \mathbb{N},$$

applying Proposition (6.1.16), we obtain $V_n^{(i)} \in U\left(z\otimes B^{(i)}\right), i = 0,1, n \in \mathbb{N}$ such that $\left(V_n^{(i)}\right)_n \in \left(z\otimes B^{(i)}\right)_\infty$and

$$\left(V_n^{(i)}a \otimes id_{B^{(i)}}\left(V_n^{(i)}\right)^*\right)_n = (un \otimes 1, i = 0,1,$$

$$\tau_{z\otimes B^{(i)}} o\ log V_n^{(i)}a \otimes id_{B^{(i)}}\left(V_n^{(i)}\right)^* u_n^* 1_{B^{(i)}} = 0, \qquad n \in \mathbb{N}$$

By the following argument, we obtain a path of unitaries $\tilde{v}_n$in $Z \otimes Z_k$ with endpoints $\Phi^{(i)}\left(V_n^{(i)}\right) \in U(Z \otimes B^{(0)} \otimes B^{(1)}), i = 0,1$ which satisfies $\tilde{v}_n a \otimes idZ_k(\tilde{v}_n^*) \approx u_n\otimes 1_{Z_k}$.

Set

$$U_{n,1} = \Phi^{(0)}\left(V_n^{(0)}\right)^* \Phi^{(1)}\left(V_n^{(1)}\right),$$

$$W_n = U_{n,1} a id_{B^{(0)}\otimes B^{(1)}}\left(U_{n,1}\right)^*, n \in \mathbb{N}$$

Then it follows that $(\left(U_{n,1}\right)_n \in \left(Z\otimes B^{(0)}\otimes B^{(1)}\right)_\infty, (w_n)_n = 1_{\left(Z\otimes B^{(0)}\otimes B^{(1)}\right)_\infty}$, and, by (i) in Proposition (6.1.16),

$$\tau_{Z\otimes B^{(0)}\otimes B^{(1)}} o\log(W_n)$$

$$= \tau olg\left(\Phi^{(1)}\left(V_n^{(1)} a \otimes id_{B^{(1)}}(V_n^{(1)})^* u_n^* \otimes 1_{B^{(1)}}\right). \Phi^{(0)}\left(V_n^{(0)} a \otimes id_{B^{(0)}}(V_n^{(0)})^* u_n^* \otimes 1_{B^{(0)}}\right)^*\right)$$

for any $n \in \mathbb{N}$.

Since $\left(U_{n,1}\right)_n \in \left(Z\otimes B^{(0)}\otimes B^{(1)}\right)_\infty$, there exist $\widetilde{U}_n \in U(C([0,1]) \otimes Z \otimes B^{(0)} \otimes B^{(1)}, n \in \mathbb{N}$ such that $\widetilde{U}_n(0) = 1, \widetilde{U}_n(1) = \widetilde{U}_{n,1}, (\widetilde{U}_n)_n \in (C([0,1]) \otimes Z \otimes B^{(0)} \otimes B^{(1)}))$ and Lip $\left(\widetilde{U}_n\right) < \pi + \varepsilon$ for some $\varepsilon > 0$. Define $\tilde{T}_n^{(j)} \in U(C([0,1]) \otimes Z \otimes B^{(0)} \otimes B^{(1)}, n \in \mathbb{N}$ by

$$\tilde{T}_n^{(j)} = \widetilde{U}_n id_{C([0,1]} a^j \otimes id_{B^{(0)}\otimes B^{(1)}}\left(\widetilde{U}_n\right)^*,$$

And $\tilde{T}_n^{(0)} = 1$. Note that

$$\tilde{T}_n^{(j)} id \otimes a \otimes id\left(\tilde{T}_n^{(j-1)}\right) = \tilde{T}_n^{(1)}, \ \ j,n \in \mathbb{N}$$

By $(\widetilde{U}_n)_n \in (C([0,1]) \otimes Z \otimes B^{(0)} \otimes B^{(1)})_\infty, (W_n)_n = 1, t\ o\ log(W_n) = 0$, and $Lip(\widetilde{U}_n) < \pi + \varepsilon$, we have that $(\tilde{T}_n^{(j)})_n \in (C([0,1]) \otimes Z \otimes B^{(0)} \otimes B^{(1)})_\infty, (\tilde{T}_n^{(j)}(1))_n = 1, t\ o\ log(\tilde{T}_n^{(j)}(1)) = jt\ o\ log(W_n) = 0$, and $Lip(\tilde{T}_n^{(j)}) < 2(p + \varepsilon), j \in \mathbb{N}$. Then, by Lemma (6.1.17), we obtain a constant $c > 0$ and $y_n^{(j)} \in U(C([0,1]^2) \otimes Z \otimes B^{(0)} \otimes B^{(1)}), j \in \mathbb{N}$ such that $\left(y_n^{(j)}\right)_n \in U(C([0,1]^2) \otimes Z \otimes B^{(0)} \otimes B^{(1)})_\infty$,

$$y_n^{(j)}(0,t) = 1, \quad y_n^{(j)}(1,t) = \tilde{T}_n^{(j)}(t), \quad t \in [0,1]$$
$$y_n^{(j)}(s,1) = exp\left(log\left(\tilde{T}_n^{(j)}(1)s\right)\right), y_n^{(j)}(s,0) = 1, s \in [0,1],$$

and $Lip\left(y_n^{(j)}\right) c, n \in \mathbb{N}$

By the Rohlin property of $a \otimes id_{B^{(0)} \otimes B^{(1)}}$ we obtain $p_m^{(l)} \in P(Z \otimes B^{(0)} \in\otimes B^{(1)})$ and $p_m^{(l)} \in U(Z \otimes B^{(0)} \in\otimes B^{(1)}), l, m \in \mathbb{N}$ such that $\left(p_m^{(l)}\right)_m \in \left(Z \otimes B^{(0)} \in\otimes B^{(1)}\right)_\infty, \left(Z_m^{(l)}\right)_m = 1$ and

$$\sum_{j=0}^{k^l-1} \left(AdZ_m^{(l)} oa\otimes id\right)^j \left(p_m^{(l)}\right) = 1.$$

Set $y_n^{(j)}(s)(t) = y_n^{(j)}(s,t), s, s \in, j, n \in \mathbb{N}$

$$\tilde{\alpha}_m^{(l)} = id_{c[0,1]} \otimes \left(AdZ_m^{(l)} oa\otimes id_{B^{(0)} \in\otimes B^{(1)}}\right), l, m \in \mathbb{N}$$

$$\widetilde{W}'_{l,m,n} = \sum_{j=0}^{k^l-1} \tilde{T}_n^{(j)} . \left(\tilde{\alpha}_m^{(l)}\right)^{j-k^l} \left(y_n^{(j)}(j / k^l)\right)^* . \left(\tilde{\alpha}_m^{(l)}\right)^j \left(id_{c[0,1]} \otimes p_m^{(l)}\right).$$

Since $\left(\left(\tilde{\alpha}_m^{(l)}\right)^j \left(id_{c[0,1]} \otimes p_m^{(l)}\right)_m \in (c[0,1]) \otimes Z \otimes B^{(0)} \in\otimes B^{(1)}\right)_\infty, j = 0,1,..,k^l - 1$ are mutually orthogonal projections, we have that $\left(\widetilde{W}'_{l,m,n}\right)_m$ is a unitary and obtain $\widetilde{W}_{l,m,n} \in U(C([0,1]) \otimes Z \otimes B^{(0)} \otimes B^{(1)}), l, m, n \in \mathbb{N}$ such that

$$\left\|(\widetilde{W}_{l,m,n})_m . id_{c[0,1]} \otimes a \otimes id\left(\widetilde{W}_{l,m,n}\right)^* - \left(\tilde{T}_n^{(1)}\right)_m\right\| < c/k^l, \quad l, m, n \in \mathbb{N}$$

Since $\left(\tilde{T}_n^{(1)}\right)_n, \left(\left(\tilde{\alpha}_m^{(l)}\right)^{j-k^l} \left(y_n^{(k^l)}(c/k^l)\right)\right)_n$, and $\left(\tilde{\alpha}_m^{(l)}\right)^j \left(1_{c[0,1]} \otimes p_m^{(l)}\right)_m \in \left(c[0,1] \otimes z \otimes B^{(0)} \in\otimes B^{(1)}\right)_\infty$, and $\|1 - W_n\| \to 0$ we obtain a slow increasing sequence $l_n, n \in \mathbb{N}$ and a fast increasing sequence $m_n \in \mathbb{N}, n \in \mathbb{N}$ such that

$$l_n \nearrow \infty, m_n \nearrow \infty,$$
$$(\widetilde{W}_{l_n,m_n,n})_n \in \left(c[0,1] \otimes z \otimes B^{(0)} \in\otimes B^{(1)}\right)_\infty,$$
$$k^{2l_n}\|1 - w_n\| \to 0, \left\|\widetilde{W}_{l_n,m_n,n} id \otimes a \otimes id\left(\widetilde{W}_{l_n,m_n,n}\right)^* - \tilde{T}_n^{(1)}\right\| < c / k^{l_n}$$

Set

$$\tilde{V}'_n = \widetilde{W}^*_{l_n,m_n,n} \widetilde{U}_n \in U\left(c[0,1] \otimes Z \otimes B^{(0)} \otimes B^{(1)}\right), n \in \mathbb{N}$$

Then it follows that $\left(\tilde{V}'_n\right)_n \in \left(c[0,1] \otimes Z \otimes B^{(0)} \otimes B^{(1)}\right)_\infty$ and

$$\left(\tilde{V}'_n di_{c[0,1]} \otimes a \otimes id_{B^{(0)} \otimes B^{(1)}} \left(\tilde{V}'_n\right)^*\right)_n$$
$$= \widetilde{W}'_{l_n,m_n,n}(1) = \sum_{j=0}^{k^{l_n}-1} \tilde{T}_n^{(1)}(1) \left(\tilde{a}_{m_n}^{(l_n)}\right)^{j-k^{l_n}} \left(y_n^{(k^{l_n})}(j/k^{l_n}, 1)\right)^* \left(\tilde{a}_{m_n}^{(l_n)}\right)^j \left(p_{m_n}^{(l_n)}\right) \approx_{\delta_n} 1,$$

Where $\delta_n = 2k^{2l_n}\|1 - W_n\|, n \in \mathbb{N}$, and then we have

$$\left(\tilde{V}'_n(1)\right)_n = \left(U_{n,1}\right)_n = \left(\Phi^{(0)} \left(V_n^{(0)}\right)^0 \Phi^{(1)} \left(V_n^{(1)}\right)\right)_n$$

Define

$$\tilde{V}_n(t) = \Phi^{(0)}\left(V_n^{(1)}\right)\tilde{V}_n'(1), t \in [0,1],$$

Then we have $\left(\tilde{V}_n\right)_n \in \left(c[0,1] \otimes Z \ \otimes B^{(0)} \otimes B^{(1)}\right)_\infty$

$$\left(\tilde{V}_n(i)\right)_n = \left(\Phi^{(i)}\left(V_n^{(i)}\right)\right)_n, i = 0,1$$

And

$$\left(\tilde{V}_n di_{c[0,1]} \otimes a \otimes id_{B^{(0)} \otimes B^{(1)}} \left(\tilde{V}_n\right)^*\right)_n = \left(1_{c[0,1]} \otimes \Phi^{(0)}\left(V_n^{(0)} a \otimes id_{B^{(0)}}\left(V_n^{(0)}\right)^*\right)\right)_n$$

$$= \left(1_{c[0,1]} \otimes u_n \otimes 1_{B^{(0)} \otimes B^{(1)}}\right)_n$$

Slightly modifying $\tilde{V}_n$ at the end points, we obtain $\tilde{V}_n \in U(Z \ \otimes Z_k)$ such that $(\tilde{v}_n)_n = \left(\tilde{V}_n\right)_n$,

$$\tilde{v}_n(i) = \Phi^{(0)}\left(V_n^{(i)}\right), i = 0,1 \qquad \left(\tilde{v}_n a \otimes id_{Z_k}(\tilde{v}_n)^*\right)_n = \left(u_n \otimes 1_{Z_k}\right)_n.$$

Finally we obtain $(v_n)_n \in Z_\infty$ which corresponds to $\tilde{v}_n \in Z \otimes Z_k$ and satisfies $v_n \alpha \ (u_n^*)_n = (v_n)_n$, by the following. By Lemma (6.1.18) and $Z_k \ \subset_{unital} Z$ we obtain a unital embedding $\Psi : Z \otimes Z_k \hookrightarrow Z^\infty$ such that $a_\infty o \, \Psi = \Psi \, o \, a \otimes id_{Z_k}$ and $\Psi\left(a \otimes 1_{Z_k}\right) = a \in Z \subset Z^\infty, a \in Z$. Let $F_n \subset Z^1, n \in \mathbb{N}$ be an increasing sequence of finite subsets of Z^1 and $\varepsilon_n > 0, n \in \mathbb{N}$ a decreasing sequence such that $\overline{\bigcup F_n} = Z^1, \varepsilon_n \searrow 0$

$$\left\|\left[\tilde{v}_n, x \otimes 1_{Z_k}\right]\right\| < \varepsilon_n, x \in F_n,$$

$$\left\|\tilde{v}_n a \otimes id_{Z_k}(\tilde{v}_n)^* - u_n \otimes 1_{Z_k}\right\| < \varepsilon_n.$$

It follows that $\|\Psi(\tilde{v}_n), x\| = \left\|\Psi\left(\left[\tilde{v}_n, x \otimes 1_{Z_k}\right]\right)\right\| < \varepsilon_n, x \in F_n$ and

Denote by $v_{n,p}\Psi \in U(Z), p \in \mathbb{N}$ components of $\Psi(\tilde{v}_n) \in U(Z^\infty)$ then we obtain an increasing sequence $p_n \in \mathbb{N} \in \mathbb{N}$ such that

$$(v_{n,p_n})_n \in Z_\infty\left(v_{n,p_n}\alpha\left(v_{n,p_n}\right)^*\right)_n = (u_n)_n.$$

Define by $u_n = v_{n,p_n}$ This completes the proof.

Corollary(6.1.20)[227]: Suppose that $a \in \mathrm{Aut}(Z)$ has the weak Rohlin property. For any finite subset F of $Z^1 a$ nd $\delta > 0$, satisfying that: for any $u \in U(Z)$ with $\|u, y\| < \delta, y \in G$, there exist $v \in U(Z)$ and $\lambda \in \mathbb{T}$ such that

$$\|va(v)^* - \lambda u\| < \|[v, x]\| < \varepsilon, x \in F$$

Proof: For $u_n \in U(Z), n \in \mathbb{N}$ with $\quad (u_n)_n \in Z_\infty, set \, \lambda_n = exp(-2\pi\sqrt{-1}\Delta_{t_z}(u_n))) \in \mathbb{T}$. Since $\Delta_{t_z}(\lambda_n u_n) = 0 \in \mathbb{R}/t\,(K_0(Z))$, by the above theorem we obtain $v_n \in U(Z), n \in \mathbb{N}$ such that $(u_n)_n \in Z_\infty$ and

$$(v_n \alpha(v_n)^*)_n = (\lambda_n u_n)_n$$

Assume that there exist a finite subset F of Z^1. and $\varepsilon > 0$ satisfying that: For any finite subset G of Z^1 and $\delta > 0$ there exists $u \in U(Z)$ with $\|[u, y]\| < \delta, y \in G$ such that if $v \in U(Z)$ and $\lambda \in \mathbb{T}$ satisfy $\|va(v)^* - \lambda u\| < \varepsilon$ then $\|[u, x]\| < \varepsilon$ for some $x \in F$. This contradicts theabove statement.

Theorem(6.1.21)[227]: Suppose that a and β are automorphisms of the Jiang–Su algebra with the weak Rohlin property. Then a and β are outer conjugate, *i.e.*, there exist an automorphism δ of Z and a unitary u in Z such that

$$a = Adu \ o \ \delta o \ \beta \ o \ \delta^{-1}.$$

Proof. By using the stability of the above form instead of Proposition 4.3 in [240] and by the Evans–Kishimoto intertwining argument in the proof of Theorem 5.1 in [240] we can give the proof. The details are as follows.

Let $\varepsilon > 0$ and let $\{x_n\}_{n\in\mathbb{N}}$be a dense sequence in Z^1. We shall construct inductively finite subsets $F_n, G_n$of $Z^1 u_n, v_n \in U(z)$, and $\delta_n > 0, n \in \mathbb{N}$ satisfying the following conditions: Set $F_0 = G_0 = \{1_Z\}, u_0 v_0 = 1_Z, a - 1 = a. \beta_0 = \beta, \delta_0 = 1.$

$$a_{2n-1} = Adu_{2n+1} o a_{2n-1}, \beta_{2n+2} = Adu_{2n+2} o \beta_{2n}, n \in \mathbb{N} \cup \{0\}$$

define

$$w_{2n} = u_{2n}\beta_{2n-2}(v_{2n})v_{2n}^*, w_{2n+1} = u_{2n+1}\beta_{2n-1}(v_{2n+1})v_{2n+1}^*$$

for $n \in \mathbb{N}$, and inductively define

$$w'_{2n} = w_{2n} Adv_{2n}(w'_{2n-2}), w'_{2n+1} = w_{2n+1} Adv_{2+1}(w'_{2n-1})$$

for $n \in \mathbb{N}$, where $w'_0 = 1, \mathrm{w}'_1 = \mathrm{w}_1$. The conditions indexed by $n \in \mathbb{N} \cup \{0\}$ are given by

(i) $F_{n+1} \supset \{x_i\}_{i=1}^{n+1} \cup \{v_n\} \cup \{w'_n\}, F_{n+1} \supset F_n,$

(ii) $G_{n+1} \supset F_{n+1} \cup G_{n+1},$

(iii) $\|Adu_{2n+1} o\, a_{2n-1}(x) - \beta_{2n}(x)\| < 2^{-1}\,\delta_{2n+1}, x \in G_{2n+1},$

(iv) $\|Adu_{2n+2} o \beta_{2n}(x) - a_{2n-1}(x) -\| < 2^{-1}\,\delta_{2n+2}, x \in G_{2n+2},$

(v) $\|v_{2n+1}a_{2n-1}(v_{2n+1})^* - u_{2n+1}\| < 2^{-2n-1}\varepsilon, \|[v_{2n+1} + 1, x]\| < 2^{-2n-1}, \varepsilon, x \in F_{2n}$

(vi) $\|v_{2n+2}\beta_{2n}(v_{2n+2})^* - u_{2n+2}\| < 2^{-2n-2}\varepsilon, \|[v_{2n+2} + 2, x]\| < 2^{-2n-2}, \varepsilon, x \in F_{2n+1}$

(vii) $\delta_{2n+1} \leq 2^{-1}\delta_{2n}$, and if $u \in U(Z)$ satisfies that $\|[u, y]\| < \delta_{2n+1}\, F2n2n + 1$ for any $y \in \beta_{2n}(G_{2n+1})$,then there exist $v \in U(Z)$ and $\lambda \in \mathbb{T}$ such that

$\|v\beta_{2n}(v)^* - \lambda u\| < 2^{-2n-2}\varepsilon, \|[v, x]\| < 2^{-2n-2}$,for any $x \in F_{2n+1}$

(viii) $\delta_{2n+2} \leq 2^{-1}\delta_{2n+1}$, and if $u \in U(Z)$ satisfies that $\|[u, y]\| < \delta_{2n+2}$, for any $y \in \alpha_{2n+1}(G_{2n+2})$, then there exist $v \in U(Z)$ and $\lambda \in \mathbb{T}$ such that

$$\|v_{2n+1}(v)^* - \lambda u\| < 2^{-2n-3}\varepsilon, \|v, x\| < 2^{-2n-3} \text{ for any } x \in F_{2n+2}$$

First, we construct F_1, satisfying (i) for $n = 0$. Assuming that we have constructed $F_n, G_n, u_n, v_n, \delta_n, n \leq 2k$, and F_{2n+1} satisfying (i) for $n \leq 2k$, (ii) for $n \leq 2k - 1$, and (iii)–(viii) for $n \leq k - 1$, we proceed as follows: Since β_{2k}has the weak Rohlin property, by Corollary (6.1.20), we obtaina finite subset $G_{2k+1}$and $\delta_{2k+1} > 0$ satisfying (ii) for $n = 2k$ and (vii) for $n = k$. Because any automorphism of the Jiang–Su algebra is approximately inner, we obtain $u_{2k+1} \in U(Z)$satisfying (iii) for $n = k$. When we obtain G_1, δ_1 and u_1 , take the same argument for $k = 0$. By (iv) for $n = k - 1$, (iii) for $n = k, G_{2k} \subset G_{2k+1}$, and $\delta_{2k} > \delta_{2k+1}$(when $k > 0$), we have that

$$\|u_{2k+1}, y\| < (\delta_{2k} + \delta_{2k+1}) < \delta_{2k}$$

for any $y \in a_{2k+1}(G_{2k})$. Then by (viii) for $n = k - 1$ we obtain $v_{2k+1} \in U(Z)$ and $\lambda_{2k+1}, \in \mathbb{T}$such that

$$\|v_{2k+1}a_{2k-1}(v_{2k+1})^*\| < 2^{-2n-1}\varepsilon, \|[v_{2k+1}, x]\| < 2^{-2n-1}$$

for $x \in F_{2k}$. When we obtain v_1, because for $F = \emptyset$ we may assume $G = \emptyset$ in Corollary (6.1.20), we obtain $v_1 \in U(Z)$ and $\lambda_1 \in \mathbb{T}$ such that $\|v_1 a(v1)^* - \lambda_1 u_1\| < 2^{-1}\varepsilon$. Since $Adu_{2k+1} = Ad\lambda_{2k+1}u_{2k+1}$, replacing u_{2k+1} we can obtain the ones which satisfy (iii) and (v) for $n = k$(for u_1 and v_1 ,wetake $k = 0$). Let F_{2k+2} satisfy (i) for $n = 2k + 1$. Similarly, by the weak Rohlin property of $a_{2k+1}$and Corollary (6.1.20), we obtain a finite subset G_{2k+2} and $d > 0$ satisfying (ii) for $n = 2k + 1$ and (viii) for $n = k$. By approximately innerness of automorphisms of the Jiang–Su algebra, (iii) for $n = k$, and (vii) for $n = k$, we obtain unitaries $u_{2k+2}$and v_{2k+2} satisfying(iv) and (vi) for $n = k$.

Finally we obtain a finite subset F_{2k+3} satisfying (i) for $n = 2k + 2$. This completes the induction.
Set $\sigma_{2n} = Ad(v_{2n}v_{2n-2} \cdots v_2)$ and $\sigma_{2n+1} = Ad(v_{2k+1}v_{2n-1} \cdots v_1)$. From (i), (v), and (vi) it follows that $\|[v_{2n+i}, v_{2n-2+i}]\| < 2^{-(2n+i)}$ for any $n \in \mathbb{N}$ and $i = 0, 1$. Then, since $\bigcup_{n\in\mathbb{N}} F_{2n}$ is dense in Z^1, we can define automorphisms of Z by

$$\tilde{\sigma}_0 = lim\, \sigma_{2n}, \qquad \tilde{\sigma}_1 = lim\, \sigma_{2n+1}.$$

Indeed for $x \in F_{2m}$ and $n > m$ it follows that $\|\sigma_{2n+2}(x) - \sigma_{2n}(x)\| < (2n + 1) \cdot 2^{-2n-2}\varepsilon$, then $\sigma_{2n}(x), n \in \mathbb{N}$is a Cauchy sequence. Similarly $\sigma_{2n+1}(x), n \in \mathbb{N}$ is also a Cauchy sequence. Thus we can define * homomorphisms $\tilde{\sigma}_0, \tilde{\sigma}_1$. Since $\|\sigma^{-1}_{2n+2+i}(x) - \sigma^{-1}_{2n+i}(x)\| < 2^{-2n-2-i}, x \in F_{2m}, n > m$, we also define *-homomorphisms $\sigma^{-1}_{2n+i}, := lim\, \sigma^{-1}_{2n+i}, i = 0, 1$, on Z. It is not so hard to see that $\tilde{\sigma}_i o\, \tilde{\sigma}_i^{-1} = id_Z = \tilde{\sigma}_i^{-1} o\tilde{\sigma}_i, i = 0, 1$.
By (i), (v) and (vi), we see that $\|w_{2n+i} - 1\| < 2^{-2n-i}\varepsilon$ and $\|[v_{2n+i}, w'_{2n-2+i}]\| 2^{-2n-2-i}\varepsilon$, then $w'_{2n-2+i}, n \in \mathbb{N}, i = 0, 1$ converge to $w_i \in U(Z), i = 0, 1$ such that $\|\tilde{w}_i - 1\| < \varepsilon$.By(iii),we have that for $x \in F_{2n+1}$

$$\|Adw'_{2n-2+i} o\sigma_{2n+1} oa\sigma^{-1}_{2n+1}(x) - Adw'_{2n} o\sigma_{2n} oa\sigma^{-1}_{2n}(x)\| < 2^{-1}\delta_{2n+1}$$

Since $\delta_n \to 0$, we conclude that

$$Ad\tilde{w}_1 o\tilde{\sigma}_1 oao\tilde{\sigma}_1^{-1} = Ad\tilde{w}_0 o\tilde{\sigma}_0 o\beta o\tilde{\sigma}_0^{-1}$$

Corollary (6.1.22)[370]: Let A^m be a unital C^*-algebra with a unique tracial state τ.
(i) For $u_1^m, u_2^m \in U(A^m)$with, $\|u_i^m - 1\| < 1/2, i = 1,2$ it follows that

$$\tau \circ log(u_1^m u_2^m) = \tau \circ log(u_1^m) + \tau \circ log(u_2^m).$$

(ii)For u_1^m, u_2^m, and $v^m \in \mathrm{U}(A^m)$with $\|u_1^m - u_2^m\| < 1/2$and $\|v^m - 1\| < 1/4$, it follows that

$$\tau \circ \log(u_1^m v^m u_2^{m*} v^{m*}) = \tau \circ \log(u_1^m u_2^{m*}).$$

Proof: (i) Let $h_i^m \in A_{sa}^m$ be such that $\exp(2\pi\sqrt{-1}h_i^m) = u_i^m, i = 1,2$, and $h_3^m \in A_{sa}^m$ be such that $\exp(2\pi\sqrt{-1}h_3^m) = u_1^m u_2^m$. Set $u^m(t) = \exp(2\pi\sqrt{-1}th_1^m) \cdot \exp(2\pi\sqrt{-1}th_2^m)$, $w(t) = \exp(2\pi\sqrt{-1}th_3^m), t \in [0,1]$. Since $\|1 - u^m(t)\| < 1, \|1 - w(t)\| < 1$, and $\|1 - w^* u^m(t)\| < 2, t \in [0,1]$, we can define $h^m \in C([0,1]) \otimes A_{sa}^m$ by $h^m(t) = \log(w^* u^m(t)), t \in [0,1]$, then u^m and ware homotopic, by $H(s,t) = w(t)\exp((1 - s)h^m(t))$with fixed endpoints $H(s,0) = 1$ and $H(s,1) = w(1)$. Hence, we have that

$$\tau o \log(u_1^m u_2^m) = 2\pi\sqrt{-1}\tau(h_3^m) = \int_0^1 \tau\,(\dot{w}w^*(t))dt = \int_0^1 \tau\,(\dot{u}^m u^{m*}(t))dt$$
$$= 2\pi\sqrt{-1}\tau(h_1^m + h_2^m) = \tau \circ log(u_1^m) + \tau \circ log(u_2^m).$$

(ii) Set $U_1 = v^{m*} u_1^m v^m u^{m*}_1, U_2 u_1^m u_2^{m*}$, then it follows that $\|U_i - 1\| < 1/2, i = 1,2$. Applying (i), since $\tau \circ log(U_1) = \tau \circ log(v^{m*}) + \tau \circ log(u_1^m v^m u_1^{m*}) = 0$ we have that $\tau \circ log(U_1 U_2) = \tau \circ log(U_1) + \tau \circ log(U_2) = \tau \circ log(U_2)$.
Corollary(6.1.23)[370]: Let B be the UHF algebra of rank k^∞, where $k \in \mathbb{N}\backslash\{1\}, \tau$ the unique tracial state of $B, \beta \in Aut(B)$, and $u_n^r \in U(B), n \in \mathbb{N}$ with $(u_n^r)_{n\in B_\infty}$. Suppose that $\beta \in Aut(B)$has the Rohlin property and

$$\sum_r \Delta_\tau(u_n^r) = 0, \qquad \text{for any } n \in \mathbb{N}.$$

Then there exist $v_n^r \in U(B), n \in \mathbb{N}$ such that$(v_n^r)_n \in B_\infty$,

$$\sum_r (v_n^r\beta(v_n^r)^*)_{r,n} = \sum_r (u_n^r)_n, \quad \sum_r \tau\circ log(v_n^r\beta(v_n^r)^*u_n^{*r}) = 0, for\ any\ n\in\mathbb{N}.$$

The following lemma was essentially proved in [233].

Proof. Because $\beta\in Aut(B)$ has the Rohlin property in [239], there exist $(v_n^r)'\in U(B), n\in N$ such that $((v_n^r)')_n\in B_\infty$, and

$$\sum_r ((v_n^r)'\beta((v_n^r)')^*)_{r,n} = \sum_r (u_n^r).$$

By the assumption and

$$\frac{1}{2\pi\sqrt{-1}}\sum_r \tau\circ\log((v_n^r)'\beta((v_n^r)')^*u_n^{*r}) + \tau(K_0(B))$$
$$= \sum_r \Delta_\tau((v_n^r)'\beta((v_n^r)')^*u_n^{*r}) = \sum_r \Delta_\tau((v_n^r)'\beta((v_n^r)')^*) - \sum_r \Delta_\tau(u_n^r)$$
$$= -\sum_r \Delta_\tau(u_n^r),$$

we have that

$$\frac{1}{2\pi\sqrt{-1}}\sum_r \tau\circ log((v_n^r)'\beta((v_n^r)')^*u_n^{*r}) + \tau(K_0(B)), n\in\mathbb{N}.$$

Since B is the UHF algebra of rank k^∞, we obtain $l_n\in\mathbb{N}$ and $m_n\in\mathbb{Z}$ such that $(m_n,k)=1$ and

$$k^{-l_n}m_n = -\frac{1}{2\pi\sqrt{-1}}\sum_r \tau\circ\log((v_n^r)'\beta((v_n^r)')^*u_n^{*r}) + \tau(K_0(B)).$$

Set $\lambda_n = exp(2\pi\sqrt{-1}k^{-l_n}m_n)$, then we have that $\lambda_n\to 1$, by $\sum_r ((v_n^r)'\beta((v_n^r)')^*u_n^{*r})_n = 1$. By the Rohlin property of $\in Aut(B)$, there exist $p_n\in P(B)$ and $z_n^r\in U(B)$, $n\in\mathbb{N}$ such that $(p_n)_n\in B_\infty$, $(z_n^r)_n = 1_{B_\infty}$, and

$$\sum_{j=0}^{k^{l_n}-1}\sum_r (\mathrm{Ad}z_n^r\mathrm{o}\beta)^j(p_n) = 1_B.$$

Define

$$\sum_r \bar{v}_n^r = \sum_{j=0}^{k^{l_n}-1}\sum_r \exp\left(2\pi\sqrt{-1}jk^{-l_n}m_n\right)\cdot(\mathrm{Ad}z_n^r\mathrm{o}\beta)^j(p_n),$$
$$\sum_r v_n^r = \sum_r (v_n^r)'\bar{v}_n^r \in U(B), n\in\mathbb{N}.$$

Taking a subsequence of $(p_n)_n$and$(z_n^r)_n$, we may suppose that $(\bar{v}_n^r)_n\in B_\infty$.
Then it follows that $(v_n^r)_n\in B_\infty$.By the definition of $\bar{v}_n^r$ we have that $\bar{v}_n^r Adz_n^r\circ\beta(\bar{v}_n^r)^* = \lambda_n$ and

$$(v_n^r\beta(v_n^r)^*u_n^{*r})_n = (v_n^r Adz_n^r\circ\beta v_n^{*r}u_n^{*r})_n = (\lambda_n(v_n^r)'\beta((v_n^r)')^*u_n^{*r})_n = 1.$$

And, by Lemma (6.1.15), we have that

$$\sum_r \tau\circ log(v_v\beta(v_n^r)^*u_n^{*r}) = \sum_r \tau\circ log(v_n^r\, Adz_n^r\circ\beta(v_n^r)^*u_n^{*r})$$

$$= \sum_r \tau \circ log(\bar{v}_n^r Adz^r n \circ \beta(\bar{v}_n^r)^* Adz_n^r \circ \beta((v_n^r)')^* u_n^{*r} (v_n^r)')$$
$$= 2\pi\sqrt{-1} k^{-l_n} m_n + \sum_r \tau \circ log(v_n^r)' \beta((v_n^r)')^* u_n^{*r} = 0, n \in \mathbb{N}.$$

Corollary (6.1.24)[370]: Suppose that $a^2 \in \mathrm{Aut}(Z)$ has the weak Rohlin property. For any finite subset F of Z^1 and $\delta > 0$, satisfying that: for any $u^m \in U(Z)$with$\|u^m, y\| < \delta, y \in G$, there exist $v^m \in U(Z)$ and $\lambda^2 \in \mathbb{T}$ such that
$$\|v^m a^2 (v^m)^* - \lambda^2 u^m\| < \|[v^m, x^m]\| < \varepsilon, x^m \in F$$
Proof: For $u_n^m \in U(Z), n \in \mathbb{N}$ with $(u_n^m)_{m,n} \in Z_\infty$, set $\lambda_n^2 = exp(-2\pi\sqrt{-1}\Delta_{t_z}(u_n^m))) \in \mathbb{T}$. Since $\Delta_{t_z}(\lambda_n^2 u_n^m) = 0 \in \mathbb{R}/t\,(K_0(Z))$, by the above theorem we obtain $v_n^m \in U(Z), n \in \mathbb{N}$ such that$(u_n^m)_{m,n} \in Z_\infty$and
$$(v_n^m \alpha (v_n^m)^*)_{m,n} = (u_n^m)_{m,n}$$
Assume that there exist a finite subset F of Z^1. and $\varepsilon > 0$ satisfying that: For any finite subset G of Z^1 and $\delta > 0$ there exists $u^m \in U(Z)$ with $\|[u^m, y]\| < \delta, y \in G$ such that if $v^m \in U(Z)$ and $\lambda^2 \in \mathbb{T}$ satisfy $\|v^m a^2 (v^m)^* - \lambda^2 u^m\| < \varepsilon$ then $\|[u^m, x^m]\| < \varepsilon$ for some $x^m \in F$. This contradicts the above statement.

Section (6.2): Automorphisms of C^*- Algebras

A major program in descriptive set theory over the last twenty-five years has been to analyze the relative complexity of classification problems by encoding these as equivalence relations on standard Borel spaces. If one can naturally parametrize the objects of a classification problem as points in a standard Borel space equipped with the relation of isomorphism, then one should expect that any reasonable assignment of complete invariants will be expressible within this descriptive framework, with the invariants being similarly parametrized. Accordingly, given equivalence relations E and F on standard Borel spaces X and Y, one says that E is Borel reducible to F if there is a Borel map $\theta : X \to Y$ such that, for all $x_1, x_2 \in X$,
$$\theta(x_1) F \theta(x_2) \Leftrightarrow x_1\, E\, x_2.$$
Borel reducibility to the relation of equality on $\mathbb{R}$ is the definition of smoothness for anequivalence relation, which was introduced by Mackey in the 1950s. In a celebrated theorem,Glimm verified a conjecture of Mackey by showing that the classification of the irreducible representations of a separable C^*-algebra is smooth if and only if the C^*-algebra is type I [286].

A much more generous notion of classification is that of Borel reducibility to the isomorphism relation on the space of countable structures of some countable language [260]. This classification by countable structures is equivalent to Borel reducibility to the orbit equivalence relation of a Borel action of the infinite permutation group S_∞ on a Polish space [354]. The isomorphism relation on any kind ofcountable algebraic structure canbe parametrized by such an orbitequivalence relation (see Example 2in [358]). Nonsmooth examples of classification by countable structures include Elliott's classification of AF algebras in terms of their ordered K-theory [161] and the Giordano-Putnam-Skau classification of minimal homeomorphisms of the Cantor set up to strong orbit equivalence [359].

A classification problem is often naturally parametrized as the orbit equivalence relation of a continuous action $G \curvearrowright X$ of a Polish group on a Polish space. Starting from the fact that every Borel map between Polish spaces is Baire measurable and hence continuous on a comeager subset,one might then aim to analyze Borel complexity in this setting by using methods of topological dynamics and Baire category. As a basic example, onecan show that the orbit equivalence relation for the action $G \curvearrowright X$ fails to be smooth whenever every orbit is dense and meager.By locally strengthening the orbit density condition in this obstruction to smoothness, Hjorth formulated the following concept of turbulence Definition (6.2.2) and proved that it obstructs classification by countable structures [260].

Definition(6.2.1)[288]: Let $G \curvearrowright X$ be an action of a topological group G on a topological space X . For $x \in X$, an open set $U \subseteq X$ which contains x, and open set $V \subseteq G$ which contains the identity element $1 \in G$, we define the local orbit $\mathcal{O}(x, U, V)$ to be the set of all $y \in U$ for which there exist $n \in \mathbb{N}$ and $g_1, g_2, \ldots, g_n \in V$ satisfying $g_k g_{k-1} \cdots g_1 x \in U$ for each $k = 1, 2, \ldots, n-1$ and $g_n g_{n-1} \cdots g_1 x = y$.

Definition(6.2.2)[288]: Let $G \curvearrowright X$ be an action of a Polish group G on a Polish space X. A point $x \in X$ is turbulent if for every U and V as in Definition(6.2.2), the closure of $\mathcal{O}(x, U, V)$ has nonempty interior. We refer to the orbit of x as a turbulent orbit. The action $G \curvearrowright X$ is saidto be turbulent if every orbit is dense, turbulent, and meager, and generically turbulent if everyorbit is meager and there exist a dense orbit and a turbulent orbit.

The definition of a turbulent orbit is sensible because one point in an orbit is turbulent if and only if all points in the orbit are turbulent. Generic turbulence is defined differently in Definition 3.20 of [260]. The equivalence of conditions (I) and (VI) in Theorem 3.21 of [260] shows that our definition is equivalent.

In [260], if $G \curvearrowright X$ is generically turbulent then for every equivalence relation F arising from a continuous action of S_∞ on a Polish space Y and every Baire measurable map $\theta: X \to Y$ such that $x_1 E\, x_2$ implies $\theta(x_1) F\, \theta(x_2)$, there exists a comeager set $C \subseteq X$ such that $\theta(x_1) F \theta(x_2)$ for all $x_1, x_2 \in C$. It follows that the orbit equivalence relation on X does notadmit classification by countable structures.

In [358] Foreman and Weiss established generic turbulence for the action of the space of measure-preserving automorphisms of a standard atomless probability space on itself by conjugation. In an analogous noncommutative setting, Kerr, Li, and Pichot showed that generic turbulence also occurs for the conjugation action $\mathrm{Aut}(R) \curvearrowright \mathrm{Aut}(R)$ where $\mathrm{Aut}(R)$ is the spaceof automorphisms of the hyperfinite II_1 factor R[267]. This raises the question of whether something similar can be said about the Borel complexity of automorphism groups in the topologicalframework of separable nuclear C^*-algebras, especially those that enjoy the regularity properties that have come to play a prominent role in the Elliott classification program [145].

For the topological analogue of an atomless probability space, namely the Cantor set X, the group $\mathrm{Homeo}(X)$ of homeomorphisms from X to itself can be canonically identified with the set of automorphisms of the Boolean algebra of clopen subsets of X (see [280]), and thus the relation of conjugacy in $\mathrm{Homeo}(X)$ is classifiable by countable structures. In particular, there is no generic turbulence, in contrast to the measurable setting. On the other hand, by [280], the relation of conjugacy in $\mathrm{Homeo}(X)$ has the maximum complexity among all equivalence relations that are classifiable by countable structures. It is thus of particular interest to determine on which side of the countable structure benchmark we can locate the automorphism groups of various noncommutative versions of zero-dimensional spaces, such as UHF algebras and the Jiang-Su algebra $\mathcal{Z}$.

We show that whenever A is $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_\infty$, a UHF algebra of infinite type, or the tensor product of a UHF algebra of infinite type and $\mathcal{O}_\infty$, then the conjugation action $\mathrm{Aut}(A) \curvearrowright \mathrm{Aut}(A)$ is generically turbulent with respect to the point-norm topology Theorem (6.2.15). We furthermore use this in the case of $\mathcal{Z}$ to prove that for every separable C^*-algebra A satisfying$\mathcal{Z} \otimes A \cong A$ (a property referred to as $\mathcal{Z}$ -stability) the relation of conjugacy on the set $\overline{\mathrm{Inn}(A)}$ ofapproximately inner automorphisms is not classifiable by countable structures Theorem (6.2.21).This class of C^*-algebras includes all of the simple nuclear C^*-algebras that fall under the scope of the standard classification results based on the Elliott invariant [145]. We thus see here an illustration of how noncommutativity tends to tilt the behaviour of a C^*-algebra more in the direction of measure theory, and not merely through the kind of "zero-dimensionality" that one frequently encounters in simple nuclear C^*-algebras. We also prove nonclassifiability by countable structures for approximately inner automorphisms of separable stable C^*-algebras Theorem (6.2.22) and of separable II_1 factors which are Mc Duff or a free product of II_1 factors Theorem (6.2.26), which includes the free group factors.

In [267], the existence of a turbulent orbit for the action $\mathrm{Aut}(\mathrm{R}) \curvearrowright \mathrm{Aut}(\mathrm{R})$ was verified by afactor exchange argument applied to the tensor product of a dense sequence of automorphismsof R. This factor exchange was accomplished by cutting into pieces which are small in trace norm and then swapping these pieces one by one to construct the required succession of small steps in the definition of turbulence. In the point-norm setting of a separable C^*-algebra, any such kind of swapping is topologically too drastic an operation if we are similarly aiming to establish turbulence, and so a different strategy is required. The novelty in our approach is to apply the exchange argument not to an arbitrary dense sequence of automorphisms but toan infinite tensor power of the tensor product shift automorphism of $A^{\otimes Z}$, which allows us tocarry out the exchange via a continuous path of unitaries in a way that commutes with the shift action. This malleability property of the tensor product shift plays an important role in Popa's deformation-rigidity theory [366] but does not seem to have appeared in the C^*-algebra context before. It is the exact commutativity of the factor exchange with the shift action that turns out to be the key for verifying turbulence. This should be compared with the kind of approximate commutativity that one's finds in a result like Lemma 2.1 of [214], which does not seem to provide enough control for our purposes. Our use of the shift also relies on the density of its conjugacy class in various situations, notably in the case of the Jiang-Su algebra $\mathcal{Z}$, for which it is a consequence of recent work of Sato [227].

To establish the other part of our turbulence theorem, namely thatevery orbit is meager, we employ a result of Rosendal whichprovides acriterion interms ofperiodic approximation for every conjugacy class in a Polish group to be meager [367] (see also page 9 of [362]). The Rokhlin lemma inergodic theorymay be seen as a prototype for this kind ofperiodic approximation, which we call the Rosendal property Definition (6.2.10). We relativize Rosendal's result in Lemma (6.2.17) so that we may use the Rosendal property in conjunction with generic turbulence to derive nonclassifiability by countable structures within the broader classes of operator algebras described above.

Throughout an undecorated $\otimes$ will denote the minimal C^*-tensor product. In fact,in all of our applications involving separable C^*-algebras at least one of the factors will be nuclear, and so there will be no ambiguity about the tensor product. We take $\mathbb{N} = \{1, 2, \ldots\}$(excluding 0). If A is a unital C^*-algebra, we denote its identity by 1_A when A must be explicitly specified.

The goal is to establish Lemma(6.2.9), which guarantees the existence of a dense turbulent orbit in $\mathrm{Aut}(A)$ for various strongly self-absorbing C^*-algebras A. This forms one component of the proof of Theorem(6.2.15), which will be completed.

Recall that a separable unital C^*-algebra$A \ncong \mathbb{C}$ is said to be strongly self-absorbing if there is an isomorphism $A \otimes A \cong A$ which is approximately unitarily equivalent to the first coordinate embedding $a \mapsto a \otimes 1$ [205]. This is a strong homogeneity property of which one consequence is$A^{\otimes \mathbb{Z}} \cong A$, which enables us to exploit the tensor product shift.

Notation(6.2.3)[288]: Let A be a separable C^*-algebra. For $\alpha \in Aut(A)$, a finite set $\Omega \subseteq A$, and $\varepsilon > 0$, we write

$$U_{\alpha,\Omega,\varepsilon} = \{\beta \in \mathrm{Aut}(A) : \|\beta(a) - \alpha(a)\| < \varepsilon \text{ for all } a \in \Omega\} .$$

These sets form a base for the point-norm topology on $Aut(A)$, under which $Aut(A)$ is a Polish group. (For some details, see Lemma 3.2 of [365].) The action $Aut(A) \curvearrowright Aut(A)$ byconjugation is continuous.

Notation(6.2.4)[288]: Let A be a unital nuclear C^*-algebra. We let $A^{\otimes \mathbb{Z}}$ be the infinite tensor product of copies of A indexed by $\mathbb{Z}$, taken in the given order. Formally, $A^{\otimes \mathbb{Z}}$ is the direct limit of thesystem

$$A \to A \otimes A \otimes A \to A \otimes A \otimes A \otimes A \otimes A \to \cdots$$

under the maps $a \mapsto 1_A \otimes a \otimes 1_A$ at each stage. A dense subalgebra is spanned by infinite elementary tensors in which all but finitely many of the tensor factors are 1_A. For $S \subseteq \mathbb{Z}$, we further write $A^{\otimes S}$ for the subalgebra of $A^{\otimes \mathbb{Z}}$ obtained as the closed linear span of all infiniteelementary tensors as above in which the tensor factors are 1_A for all indices not in S. For $m, n \in \mathbb{Z}$ with $m \leq n$, we take $A^{\otimes[m,n]} = A^{\otimes([m,n]\cap\mathbb{Z})}$. We use the analogous notation for other intervals, and for tensor powers of automorphisms as well as of algebras.

Lemma(6.2.5)[288]: Let A be a strongly self-absorbing C^*-algebra. Let γ be an automorphism of $A^{\otimes\mathbb{Z}}$, let Ω be a finite subset of $A^{\otimes\mathbb{N}}$, and let $\delta > 0$. Then there are $q \in \mathbb{N}$ and $\tilde{\gamma} \in \mathrm{Aut}\left(A^{\otimes[1,q]}\right)$ such that, with id being the identity automorphism of $A^{\otimes[q+1,\infty[}$, we have $\|(\tilde{\gamma} \otimes id)(a) - \gamma(a)\| < \delta$ for all $a \in \Omega$.

Proof: Take $q \in \mathbb{N}$ large enough that, with 1 being the identity of $A^{\otimes[q+1,\infty[}$, for every $a \in \Omega \cup \gamma(\Omega)$ there is $a^b \in A^{\otimes[1,q]}$ such that $\left\|a - a^\flat \otimes 1\right\|k < \delta/6$.

Since A is strongly self-absorbing, there is an isomorphism $\theta : A^{\otimes[1,q]} \to A^{\otimes\mathbb{N}}$ which is approx- imately unitarily equivalent to the embedding $A^{\otimes[1,q]} \hookrightarrow A^{\otimes[1,q]} \otimes A^{\otimes[q+1,\infty[} = A^{\otimes\mathbb{N}}$ given by$a \mapsto a \otimes 1$. Thus by composing θ with a suitable inner automorphism of $A^{\otimes\mathbb{N}}$ we can construct an isomorphism $\omega: A^{\otimes[1,q]} \to A^{\otimes\mathbb{N}}$ such that $\left\|\omega(a^b) - a^b \otimes 1\right\| < \delta/6$ for all $a \in \Omega \cup \gamma(\Omega)$. Set $\gamma = \omega^{-1} \circ \gamma \circ \omega \in \mathrm{Aut}\left(A^{\otimes[1,q]}\right)$. Then for every $a \in \Omega$ we have

$$\begin{aligned}\left\|\tilde{\gamma}(a^b) - \gamma(a)^b\right\| &\leq \left\|(\omega^{-1} o\, \gamma)(\omega(a^b) - a^b \otimes 1)\right\| + \left\|(\omega^{-1} o\, \gamma)(a^b \otimes 1 - a)\right\| \\ &+ \left\|\omega^{-1}(\gamma(a) - \gamma(a)^b \otimes 1)\right\| + \|\omega^{-1}(\gamma(a)^b \otimes 1) - \gamma(a)^b\| \\ &< \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{6} = \frac{2\delta}{3}\end{aligned}$$

and so

$$\begin{aligned}&\|(\tilde{\gamma} \otimes id)(a) - \gamma(a)\| \\ &\leq \left\|(\tilde{\gamma} \otimes id)(a - a^b \otimes 1)\right\| + \left\|\tilde{\gamma}(a^b) - \gamma(a)^b \otimes 1\right\| \\ &+ \left\|\gamma(a)^b \otimes 1 - \gamma(a)\right\| < \frac{\delta}{6} + \frac{2\delta}{3} + \frac{\delta}{6} = \delta,\end{aligned}$$

as desired.

Lemma(6.2.6)[288]: Let A be $Z, \mathcal{O}_2$, $\mathcal{O}_\infty$, a UHF algebra, or the tensor product of $a\; UHF$ algebra and $\mathcal{O}_\infty$. Then thetensor product shift automorphism β of $A^{\otimes\mathbb{Z}}$ has dense conjugacy class in $\mathrm{Aut}\left(A^{\otimes\mathbb{Z}}\right)$.

Proof: Consider first the case $A = Z$. Let α be an automorphism of Z, let Ω be a finite subset of Z, and let $\varepsilon > 0$. Set $M = 1 + \sup(\{\|a\| : a \in \Omega\})$. As every automorphism of Z is approximately inner (Theorem 7.6 of [177]), there is a unitary $u \in Z$ such that $\|\alpha(a) - u\beta(a)u^*\| < \varepsilon/3$ for all $a \in \Omega$. Proposition 4.4 of [227] implies that β has the weak Rokhlin property, and so by Corollary 5.6of [227] (or more precisely the simpler version omitting the quantification of finite subsets, which follows from the proof) there are a unitary $v \in Z$ and $\lambda \in \mathbb{T}$ such that $\|\lambda\mu - v\beta(v^*)\| < \varepsilon/(3M)$ (stability). Then for all $a \in \Omega$ we have

$$\|\alpha(a)-(Ad(v)\circ\beta\circ Ad(v)^{-1})(a)\| = \|\alpha(a)-v\beta(v^*)\beta(a)\beta(v)v^*\|$$
$$\leq \|\alpha(a)-u\beta(a)u^*\| + \|(\lambda u - v\beta(v^*))\|.\|\beta(a)\|.\|\bar{\lambda}u^*\|$$
$$+\|v\beta(a^*)\|.\|\beta(a)\|.\|(\lambda u - v\beta(v^*))^*\| < \frac{\epsilon}{3} + \left(\frac{\epsilon}{3M}\right)M + M\left(\frac{\epsilon}{3M}\right) \leq \epsilon.$$

Thus β has dense conjugacy class in Aut$\left(A^{\otimes Z}\right)$.

For $\mathcal{O}_2$, $\mathcal{O}_\infty$, a UHF algebra, or the tensor product of a UHF algebra and $\mathcal{O}_\infty$, we can proceed using a similar argument. Automorphisms of these C^*-algebras are well known to be approximately inner. (See for example Proposition 1.13 of [205], which shows this for every strongly self-absorbing C^*-algebra.) In the case of $\mathcal{O}_2$, $\mathcal{O}_\infty$, or the tensor product of a UHF algebra and $\mathcal{O}_\infty, \beta$, has the Rokhlin property by Theorem 1 of [244] and thus satisfies stability by Lemma 7.2 of [361]. In the case of a UHF algebra, the unital one sided tensor shift endomorphism is shown to have theRokhlin property in [355] and [363]. The Rokhlinproperty for the two sided tensor shift β follows by tensoring with 1 in front. So β satisfies stability by Theorem 1 of [360].

Definition(6.2.7)[288]: An automorphism α of a C^*-algebra A is said to be malleable if there is a point-norm continuous path $(\rho_t)_{t\in[0,1]}$ in $Aut(A\otimes A)$ such that ρ_0 is the identity, $\rho_1$is the tensor product flip, and $\rho_t\circ(\alpha\otimes\alpha)=(\alpha\otimes\alpha)\circ\rho_t$for all $t\in[0,1]$.

Lemma(6.2.8)[288]: Let A be a strongly self-absorbing C^*-algebra and let α the tensor product shift automorphism of $A\otimes Z$. Then α is malleable.

Proof: Let φ be the tensor product flip automorphism of $A\otimes A$. Since Ais strongly self-absorbing we have $A\otimes A\cong A$, and so by Theorem 2.2 of [214] we can find a norm-continuous path $(u_t)_{t\in[0,1]}$ of unitaries in $A\otimes A$ such that $u_0=1_{A\otimes A}$and $\lim_{t\to 1^-}\|u_t\, au_t^*-\varphi(a)\|=0$ for all $a\in A\otimes A$.

Define a path $(\rho_t)_{t\in[0,1]}$ in Aut$(A\otimes A)^{\otimes\mathbb{Z}}$ by setting $p_t=Ad(u_t)^{\otimes\mathbb{Z}}$ for every $t\in[0,1)$ and $\rho_t=\varphi^{\otimes\mathbb{Z}}$. Then ρ_0 is the identity. A simple approximation argument showsthat this path is point-norm continuous. Moreover, by viewing $\left((A\otimes A)^{\otimes\mathbb{Z}}\right)$ as $\left(A^{\otimes\mathbb{Z}}\right)\otimes\left(A^{\otimes\mathbb{Z}}\right)$ via the identificationthat pairs like indices, we see that $\rho_1$is the flip automorphism and $\rho_t\circ(\alpha\otimes\alpha)=(\alpha\otimes\alpha)\circ\rho_t$for all $t\in[0,1]$. Thus α is malleable.

Lemma (6.2.9)[288]: Let A be $\mathcal{Z},\mathcal{O}_2,\mathcal{O}_\infty$, a UHF algebr$a$ of infinite type, or a tensor product of a *UHF* algebra of infinite type and $\mathcal{O}_\infty$. Then there exists a dense turbulent orbit Definition (6.2.2) for the action of Aut(A) on itself by conjugation.

Proof: We follow Notation (6.2.4) throughout. Also, in this proof, for any interval S we let $id_S\in$ Aut$\left(A^{\otimes S}\right)$be the identity automorphism and let $1_S\in A^{\otimes S}$be the identity of the algebra.

Note that $A^{\otimes S}\cong A$, as all of the above C^*-algebras are strongly self-absorbing. Thus there is an automorphism β of A which is conjugate to the tensor shift automorphism of $A^{\otimes S}$. It followsfrom Lemma(6.2.8) that β is malleable. Set $\alpha=\beta^{\otimes\mathbb{N}}\in Aut\left(A^{\otimes\mathbb{N}}\right)$. By a tensor product coordinate shuffle we can view α as the shift automorphism of $\left(A^{\otimes\mathbb{N}}\right)^{\otimes\mathbb{Z}}$, and since $A^{\otimes\mathbb{N}} \cong A$ it follows that α is conjugate to β.By Lemma(6.2.6) we deduce that α has dense conjugacy class in $\mathrm{Aut}(A^{\otimes\mathbb{N}})$.Thus to establish the lemma it suces to show, given a neighbourhood U of α in $\mathrm{Aut}(A^{\otimes\mathbb{N}})$and a neighbourhood V of the identity automorphism $id_{\mathbb{N}}$ in $\mathrm{Aut}(A^{\otimes\mathbb{N}})$, that the closure of the local orbit $\mathcal{O}(\alpha, U, V)$ Definition (6.2.1) has nonempty interior.

By a straightforward approximation argument, there exist $m \in \mathbb{N}, \varepsilon > 0$, and a finite set Ω_0 in the unit ball of $A^{\otimes[1,m]}$such that, if we set

$\Omega = \{a \otimes 1_{[m+1,\infty)} : a \in \Omega_0\} \subseteq A^{\otimes N}$,

then (using Notation6.2.3) we have $U_{\alpha,\Omega,\varepsilon} \subseteq U$ and $U_{id_N,\Omega,\varepsilon} \subseteq V$.

Since β is malleable so is $\beta^{\otimes[1,m]}$, for we can rewrite $A^{\otimes[1,m]} \otimes A^{\otimes[1,m]}$as $(A \otimes A)^{\otimes[1,m]}$ by pairing like indices and then take the m-fold tensor power of a path in $Aut(A \otimes A)$witnessing the malleability of β. Thus there is a point-norm continuous path $(\rho_t)_{t\in[0,1]}$ in $A^{\otimes[1,m]} \otimes A^{\otimes[1,m]}$such that ρ_0 is the identity automorphism, $\rho_1$is the tensor product flip automorphism, and

$$\left(\beta^{\otimes[1,m]} \otimes \beta^{\otimes[1,m]}\right) \circ \rho_t = \rho_t \circ \left(\beta^{\otimes[1,m]} \otimes \beta^{\otimes[1,m]}\right) \tag{1}$$

for all $t \in [0,1]$. By point-norm continuity we can find a finite set $\mathrm{F} \subseteq A^{\otimes[1,m]} \otimes A^{\otimes[1,m]}$which is $\varepsilon/6$-densein $\{\rho_t(a \otimes 1_{[1,m]}) : a \in \Omega_0$ and $t \in [0,1]\}$.

Now choose a finite subset E_0 of the unit ball of $A^{\otimes[1,m]}$ such that for every $b \in F$ there are$\lambda_{x,y,b} \in \mathbb{C}$ for $x, y \in E_0$ with

$$\left\| b - \sum_{x,y\in E_0} \lambda_{x,y,b}\, x \otimes y \right\| < \frac{\varepsilon}{6}$$

Taking

$$M = \sup\left(\{|\lambda_{x,y,b}| : x, y \in E_0 \text{ and } b \in F\}\right),$$

for every $t \in [0,1]$and $a \in \Omega_0$ we find scalars$\lambda_{x,y,t,a} \in \mathbb{C}$with $|\lambda_{x,y,t,a}| \le M$ for $x, y \in E_0$ such

$$\left\| p_t\left(a \otimes 1_{[1,m]}\right) - \sum_{x,y\in E_0} \lambda_{x,y,t,a}\, x \otimes y \right\| < \frac{\varepsilon}{3}$$

Set

$$\varepsilon' = \frac{\varepsilon}{9(M+1)\mathrm{card}(E_0)^2} \quad \text{and } E = \{a \otimes 1_{[m+1,\infty)} : a \in E_0\} \subseteq A^{\otimes\mathbb{N}}$$

Let $W \subseteq U_{\alpha,E,\varepsilon'}$ be a nonempty open set. We will construct a continuous path $(\kappa_t)_{t\in[0,1]}$ in $\mathrm{Aut}(A^{\otimes\mathbb{N}})$ such that κ_0 is the identity automorphism, $\kappa_t \circ \alpha \circ {\kappa_t}^{-1} \in U_{\alpha,\Omega,\varepsilon}$ for all $t \in [0, 1]$, and$\kappa_1 \circ \alpha \circ {\kappa_t}^{-1} \in W$. By discretizing this path in small enough increments, this will show that$\overline{\mathcal{O}(\alpha, U, V)}$ contains $U_{\alpha,E,\varepsilon'}$and hence has nonempty interior.

A simple approximation argument provides $\gamma \in \operatorname{Aut}(A^{\otimes \mathbb{N}}), \delta > 0, q \in \mathbb{N}, with\ q > m$ and afinite set $\Upsilon_0 \subseteq A^{\otimes[1,q]}$ such that, if we set $\Upsilon = \{a \otimes 1_{[q+1,\infty)} : a \in \Upsilon_0\} \subseteq A^{\otimes \mathbb{N}}$, then we have $U\gamma, \Upsilon, \delta \ \subseteq \ W$. By Lemma(6.2.1) we may furthermore assume, increasing q if necessary,that there is an automorphism $\bar{\gamma}\ of\ A^{\otimes[1,q]}$ such that

$$\left\|\bar{\gamma} \otimes id_{[q+1,\infty)}(b) - \gamma(b)\right\| < \frac{\delta}{2} \tag{2}$$

for all $b \ \in \Upsilon$ and

$$\left\|\bar{\gamma} \otimes id_{[q+1,\infty)}(b) - \gamma(b)\right\| < \varepsilon' \tag{3}$$

for all $b \ \in \ E$.

By Lemma(6.2.6) there is an isomorphism $\theta : \ A^{\otimes[1,q]} \to \ A$ such that

$$\|(\theta^{-1} \circ \beta \circ \theta)(a) - \bar{\gamma}(a)\| < \frac{\delta}{2} \tag{4}$$

for all $a \ \in \Upsilon_0$ and

$$\left\|(\theta^{-1} \circ \beta \circ \theta)\left(x \ \otimes 1_{[q+1,\infty)}\right) - \bar{\gamma}\left(x \ \otimes 1_{[q+1,\infty)}\right)\right\| < \varepsilon' \tag{5}$$

for all $a \ \in E_0$.

Let φ be the tensor flip on $A^{\otimes[m+1,q]} \otimes A^{\otimes[m+1,q]}$. The algebra $A^{\otimes[m+1,q]} \otimes A^{\otimes[m+1,q]}$ is strongly self-absorbing and k_1-injective (since A is). So φ is strongly asymptotically inner (in the sense of Definition 1.1 (ii) of [214]) by Theorem 2.2 of [214]. Therefore there is a point-norm continuous path $(\sigma_t)_{t\in[0,1]}$of automorphisms of $A^{\otimes[m+1,q]} \otimes A^{\otimes[m+1,q]}$ such that $\sigma_0 = id$ and $\sigma_1 = \varphi$. Set

$$B = A^{\otimes[m,1]} \otimes A^{\otimes[m,1]} \otimes A^{\otimes[m+1,q]} \otimes A^{\otimes[m+1,q]}$$

and let $\psi : B \ \to \ A^{\otimes[1,q]} \ \otimes A^{\otimes[1,q]}$ be the isomorphism

$$c_1 \ \otimes \ c_2 \otimes \ d_1 \otimes \ d_2 \to \ c_1 \otimes \ d_1 \otimes c_2 \otimes \ d_2.$$

Then we have an isomorphism

$$\tau = \left(id_{[1,q]} \ \otimes \ \theta\right) \circ \ \psi : \ B \ \to \ A^{\otimes[1,q+1]} .$$

For

$$t \in [0,1], set\ \widetilde{\kappa_t} = T \circ (p_t \otimes \ \sigma_t)^{-1} \circ T^{-1} ,$$

$$\text{and define } \kappa_t = \ \widetilde{\kappa_t} \otimes \ id_{(q+2,\infty)} \in Aut\left(A^{\otimes \mathbb{N}}\right).$$

Then $(\kappa t)\ t \in [0,1]$ is a point-norm continuous path in $Aut\left(A^{\otimes \mathbb{N}}\right)$. We complete the proof by showing that

$\kappa_0 = id_{\mathbb{N}}$, that $\kappa_t \circ \ \alpha \circ \kappa_t{}^{-1} \in \ U_{\alpha,\Omega,\varepsilon}$for all $t \ \in \ [0,1]$, and that $\kappa_1 \circ \ \alpha \circ \kappa_1{}^{-1} \in \ U_{\gamma,\Upsilon,\delta}$

That $\kappa_0 = id_{\mathbb{N}}$ is obvious.

We prove that $\kappa_1 \circ \ \alpha \circ \kappa_1{}^{-1} \in \ U_{\gamma,\Upsilon,\delta}$. Let $b \ \in \ \Upsilon$. Then there is $a \ \in \Upsilon_0$ such that

$$b \ = \ a \ \otimes 1_A \otimes 1_{(q+2,\infty)} \in A^{\otimes[1,q]} \otimes \ A \ \otimes A^{\otimes(q+2,\infty)}.$$

Since ρ_1 is the tensor flip on $A^{\otimes[1,m]} \otimes A^{\otimes[1,m]}$and σ_1 is the tensor flip on $A^{\otimes[m+1,q]} \otimes A^{\otimes[m+1,q]}$,it follows that $\psi \ \circ \ (\rho_1 \otimes \sigma_1) \ \circ \ \psi^{-1}$ is the tensor flip $\varphi_q\ on\ A^{\otimes[1,q]} \otimes A^{\otimes[1,q]}$. Therefore

$$(\tilde{\kappa}_1)^{-1}(a \otimes 1_A) =$$
$$\left(id_{[1,q]} \otimes \theta\right) \circ \varphi_q \circ \left(id_{[1,q]} \otimes \theta\right)^{-1}\left(a \otimes \theta(1_{[1,q]}\right)$$
$$= 1_{[1,q]} \otimes \theta(a).$$

Continuing with similar reasoning, we conclude that

$$\left(\tilde{\kappa}_1 \circ \beta^{\otimes[1,q+1]} \circ (\tilde{\kappa}_1)^{-1}\right)(a \otimes 1_A) = (\theta^{-1} \circ \beta \circ \theta)(a) \otimes 1_A \qquad (6)$$

In the second step of the following calculation, recall that $b = a \otimes 1_A \otimes 1_{(q+2,\infty)}$, use (6.2.8)and (6.2.6) on the first term, and use (6.2.4) on the second term, getting

$$\|(\kappa_1 \circ \alpha \circ {\kappa_1}^{-1})(b) - \gamma(b)\|$$
$$\leq \left\|\left[\left(\tilde{\kappa}_1 \circ \beta^{\otimes[1,q+1]} \circ (\tilde{\kappa}_1)^{-1}\right)(a \otimes 1_A)\right] - \tilde{\gamma}(a) \otimes 1_A \otimes 1_{[q+2,\infty)}\right\|$$
$$+ \left\|\left(\tilde{\gamma}(a) \otimes id_A \otimes id_{[q+2,\infty)}\right)(b) - \gamma(b)\right\|$$
$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

Thus $\kappa_1 \circ \alpha \circ {\kappa_1}^{-1} \in U_{\gamma,Y,\delta}$, as desired.

Finally, we prove that $\kappa_1 \circ \alpha \circ {\kappa_1}^{-1} \in U_{\alpha,\Omega,\varepsilon}$. for all $t \in [0,1]$. Let $b \in \Omega$ and let $t \in [0,1]$. We need to prove that $\|(\kappa_1 \circ \alpha \circ {\kappa_1}^{-1})(b) - \alpha(b)\| < \varepsilon$. There is $a \in \Omega_0$ such that

$$b = a \otimes 1_{[m+1,q]} \otimes 1_A \otimes 1_{(q+2,\infty)} \in A^{\otimes[1,m]} \otimes A^{\otimes[m+1,q]} \otimes A \otimes A^{\otimes(q+2,\infty)}.$$

We carry out two preliminary estimates. For the first, recall that $E_0 \subseteq A^{\otimes[1,m]}$was a subsetof the unit ball chosen so that there are scalars $\lambda_{x,y} = \lambda_{x,y,t,a} \in \mathbb{C}$ with $|\lambda_{x,y}| \leq M$ for $x, y \in E_0$such that

$$\left\|\rho_t\left(a \otimes 1_{[1,m]}\right) - \sum_{x,y \in E_0} \lambda_{x,y}\, x \otimes y\right\| < \frac{\varepsilon}{3} \qquad (7)$$

We have

$$(\tilde{\kappa}_1)^{-1} a \otimes 1_{[m+1,q]} \otimes 1_A = \left(\tau \circ (\rho_t \otimes \sigma_t)\right)\left(a \otimes 1_{[1,m]} \otimes 1_{[m+1,q]} \otimes 1_{[m+1,q]}\right)$$
$$= \left(\left(id_{[1,q]} \otimes \theta\right) \circ \psi\right)\left(\rho_t\left(a \otimes 1_{[1,m]}\right) \otimes 1_{[m+1,q]} \otimes 1_{[m+1,q]}\right).$$

So

$$\left\|(\tilde{\kappa}_1)^{-1}(a \otimes 1_{[m+1,q]} \otimes 1_A - \sum_{x,y \in E_0} \lambda_{x,y} x \otimes 1_{[m+1,q]} \otimes \theta\left(y \otimes 1_{[m+1,q]}\right)\right\| < \frac{\varepsilon}{3} \quad (8)$$

Our second preliminary estimate is that for $y \in E_0$, we have

$$\left\|(\theta^{-1} \circ \beta \circ \theta)\left(y \otimes 1_{[m+1,q]}\right) - \beta^{\otimes[1,q]}\right\| < 3\acute{\varepsilon} \qquad (9)$$

To prove this, since $\gamma \in W \subseteq U_{\alpha,E,\varepsilon'}$, we have

$$\left\|\gamma\left(y \otimes 1_{[m+1,q]}\right) - \alpha\left(y \otimes 1_{[m+1,q]}\right)\right\| < \acute{\varepsilon}$$

Combine this inequality with (5) and (7) (tensoring with a suitable identity as needed) to get

$$\left\|(\theta^{-1} \circ \beta \circ \theta)\left(y \otimes 1_{[m+1,q]}\right) \otimes 1_{[q+1,\infty)[q+1,\infty)} - \alpha\left(y \otimes 1_{[q+1,\infty)}\right)\right\| < 3\acute{\varepsilon}$$

Now use

$$\alpha(y \otimes 1_{[m+1,\infty)}) = \beta^{\otimes[1,q]}(y \otimes 1_{[m+1,\infty)}) \otimes 1_{[m+1,\infty)}$$

and drop the tensor factor $1_{[q+1,\infty)}$ to get (9). From (9) and $|\lambda_{x,y}| \leq M$, $\|x\| \leq 1$, and $\|y\| \leq 1$ for $x, y \in E_0$, we then get

$$\left\| \sum_{x,y\in E_0} \lambda_{x,y} \otimes \beta^{\otimes[1,q]}(x \otimes 1_{[m+1,\infty)}) \otimes (\theta^{-1} \circ \beta \circ \theta)(y \otimes 1_{[m+1,q]}) - \sum_{x,y\in E_0} \lambda_{x,y} \otimes \beta^{\otimes[1,q]}(x \otimes 1_{[m+1,\infty)}) \otimes \beta^{\otimes[1,q]}(y \otimes 1_{[m+1,q]}) \right\| \leq 3Mcard(E_0)^2 \acute{\varepsilon} < \frac{\varepsilon}{3} \tag{10}$$

We are now ready to show that $\|(\kappa_t \circ \alpha \circ {\kappa_t}^{-1})(b) - \alpha(b)\| < \varepsilon$. We calculate (justificationsgiven afterwards):

$$(\kappa_t \circ \alpha \circ {\kappa_t}^{-1})(b) - \alpha(b) \approx_{\epsilon/3} \left(\sum_{x,y\in E_0} \lambda_{x,y}\, \beta^{\otimes[1,q]}(x \otimes 1_{[m+1,\infty)}) \otimes (\beta \circ \theta)(y \otimes 1_{[m+1,q]}) \right) \otimes 1_{[q+2,\infty)}$$

$$= \left((id_{[1,q]} \otimes \theta) \circ \psi \circ (\rho_t \otimes \sigma_t)^{-1} \circ \psi^{-1} \right) \left(\sum_{x,y\in E_0} \lambda_{x,y}\, \beta^{\otimes[1,q]}(x \otimes 1_{[m+1,\infty)}) \otimes (\theta^{-1} \circ \beta \circ \theta)(y \otimes 1_{[m+1,q]}) \right) \otimes 1_{[q+2,\infty)}$$

$$\approx_{\epsilon/3} (id_{[1,q]} \otimes \theta) \circ \psi \circ (\rho_t \otimes \sigma_t)^{-1} \circ \psi^{-1}) \left(\sum_{x,y\in E_0} \lambda_{x,y}\, \beta^{\otimes[1,q]}(x \otimes 1_{[m+1,\infty)}) \otimes \beta^{\otimes[1,q]}(y \otimes 1_{[m+1,q]}) \right) \otimes 1_{[q+2,\infty)}$$

$$(id_{[1,q]} \otimes \theta) \circ \psi \circ (\rho_t \otimes \sigma_t)^{-1}) \left(\sum_{x,y\in E_0} \lambda_{x,y}\, \beta^{\otimes[1,m]}(x) \otimes \beta^{\otimes[1,m]}(y) \otimes 1_{[m+1,q]} \otimes 1_{[m+1,q]} \right) \otimes 1_{[q+2,\infty)}$$

$$= \left((id_{[1,q]} \otimes \theta) \circ \psi \right) \left(\sum_{x,y\in E_0} \lambda_{x,y} (\beta^{\otimes[1,m]} \otimes \beta^{\otimes[1,m]})({\rho_t}^{-1}(x \otimes y)) \otimes 1_{[m+1,q]} \otimes 1_{[m+1,q]} \right) \otimes 1_{[q+2,\infty)}$$

$$= \left((id_{[1,q]} \otimes \theta) \circ (\beta^{\otimes[1,m]} \otimes \beta^{\otimes[1,m]}) \right) \circ \psi \circ ({\rho_t}^{-1} \otimes id_{[m+1,q]} \otimes id_{[m+1,q]})$$

$$\left(\sum_{x,y\in E_0} \lambda_{x,y} x \otimes y \otimes 1_{[m+1,q]} \otimes 1_{[m+1,q]}\right) \otimes 1_{[q+2,\infty)}$$

$$\approx_{\epsilon/3} \left(\left(id_{[1,q]} \otimes \theta\right) \circ \left(\beta^{\otimes[1,q]} \otimes \beta^{\otimes[1,q]}\right) \circ \psi\right) \left(a \otimes 1_{[1,m]} \otimes 1_{[m+1,q]} \otimes 1_{[m+1,q]}\right) \otimes 1_{[q+2,\infty)}$$

$$= \beta^{\otimes[1,q]}\left(a \otimes 1_{[m+1,q]}\right) \otimes 1_{[q+2,\infty)}$$

$$= \alpha(b).$$

The first step follows from (8) and $\alpha = \beta^{\otimes\mathbb{N}}$. The second step is the definition of $\tilde{\kappa}_t$. The third follows from (10). The fourth is the definition of ψ and $\beta^{\otimes[m+1,q]}(1) = 1$. For the fifth, we use

$$\left(\beta^{\otimes[1,m]} \otimes \beta^{\otimes[1,m]}\right) \circ \rho_t^{-1} = \rho_t^{-1} \circ \left(\beta^{\otimes[1,m]} \otimes \beta^{\otimes[1,m]}\right),$$

which follows from (1). The sixth step uses the definition of ψ and the relation $\beta^{\otimes[m+1,q]}(1) = 1$. The seventh step follows from (7), the eighth is easy, and the last step is $\alpha = \beta^{\otimes\mathbb{N}}$.

For any unital - algebra A, we denote its unitary group by $U(A)$, and equip it with the norm topology.

To establish turbulence for the action $U(A) \curvearrowright Aut(A)$ we proceed as follows. Observe that the orbits are just translates of the group Inn(A) of inner automorphisms. As $Inn(A)$a non- closed Borel subgroup of $Aut(A)$ [297], it follows from Pettis's theorem (see [259]) that $Inn(A)$ is meager in $Aut(A)$. Moreover, $Inn(A)$ is dense in $Aut(A)$ by Proposition 1.13 of [205]. It follows that every orbit is dense and meager. It thus remains to show, given $\alpha \in Aut(A)$, a neighbourhood U of α in $Aut(A)$, and a neighbourhood$V$ of 1 in $U(A)$, that the local orbit $\mathcal{O}(\alpha, U, V)$ is somewhere dense.

To this end, we may assume that U is of the form $U_{\alpha\Omega\varepsilon}$ as in Notation 6.2.1 for some finite set $\Omega \subseteq A$ and $\varepsilon > 0$, and that $V = \{u \in U(A): \|u - 1\| < \varepsilon\}$. Write $U_0(A)$ for the path connected component of the identity in the unitary group of A. By Lemma 2.1 of [214], there are a finite set $\Upsilon \subseteq A$ and $\delta > 0$ such that if w is a unitary in $U_0(A)$ satisfying $\|[w,x]\| < \delta$for all $x \in \Upsilon$,then there is a continuous path $(w_t)_{t\in[0,1]}$ of unitaries in $U_0(A)$ such that $w_0 = w$, $w_1 = 1$, and $\|[w_t,x]\| < \varepsilon$for all $x \in \alpha(\Omega)$ and $t \in [0,1]$. To complete the argument we will show that the open set $U_{\alpha,\alpha-1(\Upsilon),\delta}$ is contained in the closure of $\mathcal{O}(\alpha, U, V)$. So let $\beta \in U_{\alpha,\alpha^{-1}(\Upsilon)}, \delta$ and let W be an open neighbourhood of β contained in U. By Theorem 3.1 of [369], the algebra A is automatically$\mathcal{Z}$- stable. In particular, (see Remark 3.3 of [369]), it is K_1-injective, so Proposition 1.13 of [205] applies. Thus there is $u \in U_0(A)$ such that $Ad(u) \circ \alpha \in W \subseteq U_{\alpha,\alpha^{-1}(\Upsilon),\delta}$. In particular, $Ad(u) \circ \alpha \in U_{\alpha,\alpha^{-1}(\Upsilon),\delta}$, and so by our choice of Υ and δ there is a continuous path $(u_t)_{t\in[0,1]}$ of unitaries in $U_0(A)$ such that $u_0 = u$, $u_1 = 1$, and $\|[u_t,x]\| < \varepsilon$for all $x \in \alpha(\Omega)$ and $t \in [0,1]$.

This last condition is the same as saying that $Ad(u_t) \circ \alpha \in U_{\alpha,\Omega,\varepsilon}$ for all $t \in [0,1]$. We can now discretize the path $(u_t)_{t\in[0,1]}$ in small enough increments to verify the membership of β in $\mathcal{O}(\alpha, U, V)$. We conclude that $U_{\alpha,\alpha^{-1}(\Upsilon),\delta}$ is contained in the closure of $\mathcal{O}(\alpha, U, V)$, as desired.
Implies that automorphisms of strongly self-absorbing C^*-algebras are not clas- sifiable up to unitary equivalence by countable structures. This consequence is proved using different methods in [278], in much greater generality (for separable C^*-algebras which do not have continuous trace).
With the aim of completing the proof of Theorem(6.2.15), we now concentrate on verifying the meagerness of orbits condition in the definition of generic turbulence. For this we will employ a result of Rosendal that gives a criterion in terms of periodic approximation for every conjugacy class in a Polish group to be meager [367]. As we will later relativize this result in Lemma(6.2.17) for applications, it will be convenient to abstract the relevant periodic approximation property into a definition.
Definition(6.2.10)[288]: We say that a Polish group Ghas the Rosendal property if for every infinite set $I \subseteq \mathbb{N}$ and neighbourhood V of 1 in $G$the set.

$$\{g \in G : \text{there is } n \in I \text{ such that } g^n \in V\}$$

is dense. Rosendal's result [367] can now be formulated as follows.
Lemma(6.2.11)[288]: Let G be a nontrivial Polish group with the Rosendal property. Then every con- jugacy class in G is meager.

For a unital C^*-algebra A we write $U_0(A)$ for the path connected component of the identity in the unitary group $U(A)$ of A, and $\mathrm{Inn}_0(A)$ for the normal subgroup of $\mathrm{Aut}(A)$ consisting of all automorphisms of A of the form $\mathrm{Ad}(u)$ for some $u \in U_0(A)$.
Lemma(6.2.12)[288]: Let A be a separable unital C^*-algebra with real rank zero such that $\mathrm{Inn}_0(A)$is dense in $\mathrm{Aut}(A)$. Then $\mathrm{Aut}(A)$has the Rosendal property.
Proof: Let I be an infinite subset of $\mathbb{N}$. Set

$$S = \{\varphi \in \mathrm{Aut}(A) \text{: there is } n \in I \text{ such that} \varphi^n = id_A\}.$$

It suffices to prove that S is dense. Let $\alpha \in \mathrm{Aut}(A)$, let $\Omega \subseteq A$be finite, and let $\varepsilon > 0$. It suffices to show (following Notation (6.2.3)) that $S \cap U_{\alpha,\Omega,\varepsilon} \neq \emptyset$. Set $M = 1 + \sup(\{\|a\| : a \in \Omega\})$.
As real rank zero is equivalent to the density in $U_0(A)$ of the unitaries in $U_0(A)$ with finite spectrum [364], the density of $\mathrm{Inn}_0(A)$ in $\mathrm{Aut}(A)$ implies the existence of a unitary u with finite spectrum such that $\|\alpha(a) - uau^*\| < \varepsilon/2$ for all $a \in \Omega$. Since u has finite spectrum, there are $k \in \mathbb{N}$, projections $p_1, p_2, \ldots, p_k \in A$, and $\theta_1, \theta_2, \ldots, \theta_k \in [0,1)$ such that $u = \sum_{j=1}^{k} e^{2\pi i\theta_j/n} p_j$.

Choose $n \in I$ such that $n > 8\pi M/\varepsilon$, and for $j = 1,2,\ldots,k$choose $m_j \in \{0,1\ldots,n-1\}$ such that $|\theta_j - m_j/n| < 1/n$. Set $v = \sum_{j=1}^{k} e^{2\pi i m_j/n} p_j$. Then $v^n = 1$ and so $Ad(v)^n = id$. Moreover, since

$$\|u-v\| \leq \sup_{1\leq j\leq k} 2\pi \left|\theta_j - \frac{m_j}{n}\right| \leq \frac{2\pi}{n} < \frac{\varepsilon}{4},$$

We have, for every $\alpha \in \Omega$

$$\|\alpha(a) - vav^*\| \leq \|\alpha(a) - uau^*\| + \|u - v\| . \|a\| . \|u^*\| + \|v\| . \|a\| \cdot \|(u - v)^*\|$$
$$< \frac{\varepsilon}{3} + \left(\frac{\varepsilon}{3M}\right) M + M\left(\frac{\varepsilon}{3M}\right) = \varepsilon.$$

Thus $Ad(v) \in U_{\alpha,\Omega,\varepsilon,}$ as required.

Lemma(6.2.12) shows that $\mathrm{Aut}(A)$ has the Rosendal property when A is $\mathcal{O}_2, \mathcal{O}_\infty$, a UHF algebra, or the tensor product of a UHF algebra and $\mathcal{O}_\infty$, but cannot be applied to $\mathcal{Z}$ since $\mathcal{Z}$ does not have real rank zero. Indeed the only projections in $\mathcal{Z}$ are 0 and 1. Nevertheless we can useanother argument based on the shift automorphism.

Lemma(6.2.13)[288]: $\mathrm{Aut}(\mathcal{Z})$has the Rosendal property.

Proof: Let I be an infinite subset of $\mathbb{N}$. As in the proof of Lemma(6.2.12), we actually show that automorphisms with orders in I are dense. Thus set

$$S = \{\varphi \in \mathrm{Aut}(A): \text{there is } n \in I \text{ such that } \varphi^n = id_A\}$$

let $\alpha \in \mathrm{Aut}(A)$, let $\Omega \subseteq A$ be finite, and let $\varepsilon > 0$. We show that $S \cap U_{\alpha,\Omega,\varepsilon} \neq \emptyset$.

Let β be the tensor shift automorphism of $\mathcal{Z}^{\otimes\mathbb{Z}}$. By Lemma(6.2.6) there is an isomorphism $\gamma: \mathcal{Z}^{\otimes\mathbb{Z}} \to \mathcal{Z}$ such that $\|(\gamma \circ \beta \circ \gamma^{-1})(a) - \alpha(a)\| < \varepsilon/3$ for all $a \in \Omega$. By the definition of the infinite tensor product, there are $m \in \mathbb{N}$ and a finite set

$$\Upsilon \subseteq 1 \otimes \mathcal{Z}^{\otimes[-\mathrm{m},\mathrm{m}]} \otimes 1 \subseteq \mathcal{Z}^{\otimes\mathbb{Z}}$$

such that for every $a \in \Omega$ there is $b \in \Upsilon$ with $\|\gamma^{-1}(a) - b\| < \varepsilon/3$. Choose $n \in I$such that $n \geq 2m + 2$. Let $\kappa \in \mathrm{Aut}\left(\mathcal{Z}^{\otimes[-m,n-m-1]}\right)$ be the forwards cyclic tensor shift automorphism, which for $x_{-m}, x_{-m+1}, \dots, x_{n-m-1} \in \mathcal{Z}$ satisfies

$$\kappa(x_{-m} \otimes x_{-m+1} \otimes \cdots \otimes x_{n-m-2} \otimes x_{n-m-1})$$
$$= x_{n-m-1} \otimes x_{-m} \otimes x_{-m+1} \otimes \cdots \otimes x_{n-m-2}$$

Then $\kappa^n = id$.

Let then

$$k^n = id \otimes \kappa \otimes id \in Aut\left(\mathcal{Z}^{\otimes(-\infty,-m-1]} \otimes \mathcal{Z}^{\otimes[-m,n-m-1]} \otimes \mathcal{Z}^{\otimes[n-m,\infty)}\right)$$
$$= Aut\left(\mathcal{Z}^{\otimes\mathbb{Z}}\right).$$

Then $\psi^n = id$ (so that $\gamma \circ \psi \circ \gamma^{-1} \in S$) and $\psi(b) = \beta(b)$ for all $b \in 1 \otimes \mathcal{Z}^{\otimes[-\mathrm{m},\mathrm{m}]} \otimes 1 \subseteq \mathcal{Z}^{\otimes\mathbb{Z}}$.

Now let $a \in \Omega$. Choose $b \in \Upsilon$ such that $\|\gamma^{-1}(a) - b\| < \varepsilon/3$. Using $\psi(b) = \beta(b)$, we get

$$\|(\gamma \circ \psi \circ \gamma^{-1})(a) - \alpha(a)\|$$
$$\leq \|(\gamma \circ \psi)(\gamma^{-1}(a) - b)\| + \left\|(\gamma \circ \beta)\left(b - \gamma^{-1}(a)\right)\right\| + \|(\gamma \circ \beta \circ \gamma^{-1})(a) - \alpha(a)\|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus $\gamma \circ \psi \circ \gamma^{-1} \in U_{\alpha,\Omega,\varepsilon}$, which establishes the desired density.

From Lemmas(6.2.11),(6.2.12) and(6.2.13) we obtain:

Lemma(6.2.14)[288]: Let A be $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_\infty$, a UHF algebra, or the tensor product of a UHF algebra and $\mathcal{O}_\infty$. Then every conjugacy class in $\mathrm{Aut}(A)$ is meager.
Lemmas (6.2.4) and(6.2.8) together yield the following.
Theorem(6.2.15)[288]: Let A be $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_\infty, a$ UHF algebra of infinite type, or the tensor product of a UHF algebra of infinite type and $\mathcal{O}_\infty$. Then the conjugation action $\mathrm{Aut}(A) \curvearrowright \mathrm{Aut}(A)$is generically turbulent.
Consider a standard atomless probability space (X, μ) and the Polish group $\mathrm{Aut}(X, \mu)$ of measure-preserving transformations of X under the weak topology. In [358] Foreman and Weiss showed that restriction of the conjugation action $\mathrm{Aut}(X, \mu) \curvearrowright \mathrm{Aut}(X, \mu)$ to the G_δ subset of essentially free ergodic automorphisms is turbulent and not merely generically turbulent. The essentially free automorphisms are precisely those which satisfy the Rokhlin lemma. The analogue of freeness for automorphisms of $\mathcal{Z}$ is the property that every nonzero power of the automorphism is strongly outer, which is equivalent to the weak Rokhlin property [227]. The set $WRok(A)$ of automorphisms of $\mathrm{Aut}(\mathcal{Z})$ with the weak Rokhlin property is easily seen to be a G_δ set, and it is dense by Lemma(6.2.6) as the tensor product shift automorphism of $\mathcal{Z}$ is strongly outer. In analogy with the Foreman-Weiss result we ask the following.
Problem(6.2.16)[288]: Is the conjugation action $\mathrm{Aut}(\mathcal{Z}) \curvearrowright WRok\ (\mathcal{Z})$ turbulent?

Using the stability of automorphisms of $\mathcal{Z}$ with the weak Rokhlin property [227], it can be shown as in the proof of Lemma(6.2.6) that any automorphism of $\mathcal{Z}$ with the weak Rokhlin property has dense conjugacy class in $Aut(\mathcal{Z})$. So the question of turbulence for the action $Aut(\mathcal{Z}) \curvearrowright WRok\mathcal{Z}$ amounts to the problem of whether every orbit in $WRok\ (\mathcal{Z})$is turbulent.

We can also ask the same question for the conjugation action $Aut(A) \curvearrowright WRok(A)$ on the set of automorphisms satisfying the Rokhlin property when A is any one of the other C^*-algebras in Theorem(6.2.15).

We prove Theorem(6.2.21):for a separable $\mathcal{Z}$-stable C^*-algebra A, the orbit equivalence relation of the conjugation action $\mathrm{Aut}(A) \curvearrowright \overline{\mathrm{Inn}(A)}$ is not classifiable by countable structures.
Lemma(6.2.17)[288]: Let G and H be Polish groups such that G has the Rosendal property Definition (6.2.10). Let $\varphi : G \to H$ be a continuous homomorphism such that $\varphi(G) \neq \{1_H\}$. Let E be an equivalence relation on G such that for every infinite set $I \subseteq \mathbb{N}$ the set

$$Q_1 = \left\{ \begin{matrix} g \in G\text{: there is } a \text{ strictly increasing sequence} (k_\mathrm{n})_{n=1}^{\infty} \text{in } I \text{ such that} \\ \varphi(g)^{\mathrm{k}n} \to 1 \end{matrix} \right\}$$

is E-invariant. Then every equivalence class of E that is dense in G is meager. In particular E does not have a comeager class.
Proof: Let $I \subseteq \mathbb{N}$ be infinite. We claim that $Q_1$is comeager. To prove the claim, choose acountable base $(V_\mathrm{n})_{n=1}^{\infty}$ of open neighbourhoods of 1_H in H such that $V_1 \supseteq V_2 \supseteq \cdots$. For $n \in \mathbb{N}$ define

$$Q_{1,n} = \{g \in G: \text{there is } k \in 1 \text{ such tha } k \geq n \text{ and } \varphi(g)^{\mathrm{k}} \in V_n\}$$

Then $Q_{1,n}$ is open and contains the set

$$\{g \in G: \text{there is } k \in 1\ \{1,2,3,\dots,n-1\} \text{ such that } \varphi(g)^{\mathrm{k}} \in V_n\}$$

which is dense in $G$by the Rosendal property. Since $Q_1 = \bigcap_{n=1}^{\infty} Q_{1,n}$, the claim follows.

Now let C be an equivalence class of E that is dense in G, and suppose that C is not meager.Let $g \in C$. Then for every infinite $I \subseteq N$ the set Q_1, being comeager and E-invariant, contains C. Therefore every subsequence $\left(\varphi(g)^{\mathrm{l}_n}\right)_{n=1}^{\infty}$of $(\varphi(g)^{\mathrm{n}})_{n=1}^{\infty}$ in turn has a subsequence whichconverges to 1_H . It follows that $\varphi(g)^{\mathrm{n}} \to 1_H$. Since also $\varphi(g)^{\mathrm{n+1}} \to 1_H$, we conclude that $\varphi(\mathrm{g}) = 1_H$. Thus $\varphi^{-1}(\{1_H\})$ contains C and hence is dense in G Since φ is continuous, we conclude that $\varphi^{-1}(\{1_H\}) = G$. This contradicts our hypothesis that $\varphi(\mathrm{G}) \neq \{1_H\}$.

We let S_∞ denote the set of all permutations of $\mathbb{N}$ (equivalently, of any countable set), which is a Polish group in a standard way. Also, for an action $G \curvearrowright X$of a group G on a set X,we write E_G^X for the orbit equivalence relation on X.

Definition(6.2.18)[288]: (Definition 3.6 of [260]). Let E be an equivalence relation on a Polish Space X, and let F be an equivalence relation on a Polish Space Y. A Baire homomorphism from E to F is a Baire measurable function $\varphi : X \to Y$such that whenever $x_1, x_2 \in X$satisfy $x_1 E\, x_2$, then $\varphi(x_1) F \varphi(x_2)$. We say that E is generically F-ergodic if for any Baire homomorphism $\varphi: X \to Y$there is a comeager set $C \subseteq X$such that the image of C under $\varphi$is contained in a single F -equivalence class.

From the point of view of applications, the following lemma is the main result in [260], although it is not explicitly stated there.

Lemma(6.2.19)[288]: Let$G \curvearrowright X$ be a continuous action of a Polish group G on a Polish space X, and let E the corresponding orbit equivalence relation. If the action is generically turbulent, thenis generically$E_{S_\infty}^Y$-ergodic for every Polish S_∞-space Y .

Proof: By condition (VII) in Theorem 3.21 of [260], there is a $G -$ invariant dense$G_\delta -$ set in Xsuch that the restriction of the action to this set is turbulent. It is clearly enough to show generic$E_{S_\infty}^Y$-ergodicity for this subset. Apply Theorem 3.18 of [260].

Lemma(6.2.20)[288]: Let G be a Polish group with the Rosendal property such that the relation ofconjugacy in G is generically $E_{S_\infty}^Y$-ergodicfor every Polish S_∞-space Y. Let H be a Polishgroup and let $\varphi: G \to H$ a continuous homomorphism such that $\varphi(\mathrm{G}) \neq \{1_H\}$. Let F be the equivalence relation on $\varphi(\mathrm{G})$given by xFy if there is $h \in H$for which $y = hxh^{-1}$. Then F is not classifiable by countable structures.

Proof: Suppose to the contrary that F is classifiable by countable structures. Then there is a space $\mathcal{Z}$of countable structures for a countable language and a Borel map $\psi : G \to \mathcal{Z}$such that, with $\cong$ denoting the orbit equivalence relation of the canonical action $S_\infty \curvearrowright \mathcal{Z}$, we have xFy if and only if $\psi(x) \cong \psi(y)$. (See Definition 2.37 and Definition 2.37 of [260].) Let E be the equivalence relation on Gsuch that sEtif there is $h \in H$for which

$\varphi(t) = h\varphi(s)h^{-1}$. By hypothesis the relation of conjugacyin $:G$ is generically $E_{S_\infty}^{Y}$ -ergodicand so there is a comeagersubset C of G such that for all $s,t \in C$ we have $(\psi \circ \varphi)(s) \cong (\psi \circ \varphi)(t)$ and hence sEt.

Now let $s,t \in G$ satisfy sEt and let $(k_n)_{n=1}^{\infty}$ be a strictly increasing sequence in N such that $\varphi(s)^{k_n} \to 1$. By the definition of E, there is $h \in H$ such that $\varphi(t) = h\varphi(s)h^{-1}$. Then

$$\varphi(t)^{k_n} = h\varphi(s)h^{-1} \to 1.$$

This shows that for every infinite $I \subseteq N$ the set Q_1 in Lemma(6.2.17) is E-invariant. We apply that lemma to deduce that E does not have a comeager class, contradicting the comeagerness of C. We thus conclude that F is not classifiable by countable structures.

Clearly in the statement of Lemma(6.2.20) one can replace $\varphi(G)$ with $\varphi(X)$ for any comeager Borel subset X of G that is invariant under conjugation.

For a C^*-algebra A we write $\mathrm{Inn}(A)$ for the set of inner automorphisms of A, and note that the closure $\overline{\mathrm{Inn}(A)}$ is a normal subgroup of $\mathrm{Aut}(A)$.

Theorem(6.2.21)[288]: Let A be a separable $\mathcal{Z}$-stable C^*-algebra. Then the orbit equivalence relation of the conjugation action $\mathrm{Aut}(A) \curvearrowright \overline{\mathrm{Inn}(A)}$ is not classifiable by countable structures.

Proof: Identify A with $\mathcal{Z} \otimes A$. The map $\alpha \mapsto \alpha \otimes \mathrm{id}_A$ is a continuous homomorphism from $\mathrm{Aut}(\mathcal{Z})$ onto a closed subgroup of $Aut(\mathcal{Z} \otimes A)$. Since all automorphisms of $\mathcal{Z}$ are approximatelyinner (Theorem 7.6 of [177]), its image is contained in $\overline{\mathrm{Inn}(A)}$. By Lemma(6.2.13) the group $\mathrm{Aut}(\mathcal{Z})$ has the Rosendal property, and by Lemma(6.2.9) and Lemma (6.2.19) the orbit equivalence relation ofthe conjugation action $\mathrm{Aut}(\mathcal{Z}) \curvearrowright \overline{\mathrm{Inn}(\mathcal{Z})}$ is generically $E_{S_\infty}^{Y}$ -ergodicfor every Polish S_∞ -space Y. We thus obtain the conclusion by applying Lemma (6.2.20).

Using Theorem 4.17 of [365] and the fact that the automorphism constructed in the proof of Lemma(6.2.9) has the tracial Rokhlin property [365], we can furthermore deduce from the proof of Theorem(6.2.21) that if A is a simple separable unital infinite-dimensional C^*-algebra with tracial rank zero, then the approximately inner automorphisms of A with the tracial Rokhlin property are not classifiable by countable structures up to conjugacy. Similarly, using Theorem 5.13 of [365] we can conclude that if A is a separable unital $\mathcal{O}_2$ -stable C^* -algebra, then the approximately inner automorphisms of A with the Rokhlin property are not classifiable by countable structures up to conjugacy.

In the particular case when A is the Cuntz algebra of $\mathcal{O}_2$, [285] provides further information about the complexity of the orbit equivalence relation of the conjugation action $Aut(\mathcal{O}_2) \curvearrowright Aut(\mathcal{O}_2)$: Such equivalence relation is not Borel as a subset of $Aut(\mathcal{O}_2) \times Aut(\mathcal{O}_2)$. Moreover if $\mathcal{C}$ is any class of countable structure such that the relation $\cong_e$ of isomorphism ofelements of $\mathcal{C}$ is Borel, then $\cong_e$ is Borel reducible to the relation of conjugacy of automorphismsof $\mathcal{O}_2$. The same conclusions hold if one considers the relation of cocycle conjugacy of auto- morphisms of $\mathcal{O}_2$. (Recall that two automorphisms α, β of a unital C^*-algebra A are cocycle conjugate if there is a unitary element u of A such that $\mathrm{Ad}(u) \circ \alpha$ and β are conjugate.)

Fix a separable infinite dimensional Hilbert $\mathcal{H}$, and let $\mathcal{K}$ be the C^*-algebra of compact operators on $\mathcal{H}$. Recall that a C^*-algebra A is said to be stable if $\mathcal{K} \otimes A \cong A$. Here we show using Lemma(6.2.22) that if A is a stable C^*-algebra then the orbit equivalence relation of theconjugation action $\mathrm{Aut}(A) \curvearrowright \overline{\mathrm{Inn}(A)}$ is not classifiable by countable structures.

Lemma(6.2.22)[288]: The unitary group $U(\mathcal{H})$ has the Rosendal property.

Proof: The proof is like part of the proof of Lemma(6.2.12). Set

$$S = \{u \in U(\mathcal{H}) \text{: there is } n \in I \text{ such that } u^n = 1\}.$$

It suffices to prove that S is dense. Let $v \in U(\mathcal{H})$ and let $\varepsilon > 0$. Choose $n \in I$ such that $2\pi/n < \varepsilon$. Let S^1denote the unit circle in $\mathbb{C}$. Let $f : S^1 \to S^1$ be the Borel function which,for $k = 0, 1, \ldots, n-1$, takes the value $\exp(2\pi ik/n)$ on the arc $\{exp(2\pi i\theta)\colon\ k/n \le \theta < k + 1/n\}$. Then$u = f(v) \in U(\mathcal{H})$ satisfies $u^n = 1$, so that $u \in S$ and $\|u - v\| \le 2\pi/n < \varepsilon$.

Theorem(6.2.23)[288]: Let A be a separable stable C^*-algebra. Then the orbit equivalence relation of the conjugation action $\mathrm{Aut}(A) \curvearrowright \overline{\mathrm{Inn}(A)}$ is not classifiable by countable structures.

Proof: Identify A with $\mathcal{K} \otimes A$. The map $\alpha \mapsto \alpha \otimes id_A$ is a continuous homomorphism from $Aut(\mathcal{K})$ onto a closed subgroup of $\mathrm{Aut}(\mathcal{K} \otimes A)$. Since every automorphism of $\mathcal{K}$ is inner, this subgroup is contained in $\overline{\mathrm{Inn}(A)}$. By Theorem 6.1 of [263] the conjugation action $U(\mathcal{H}) \curvearrowright U(\mathcal{H})$ is generically turbulent and hence the corresponding orbit equivalence relation is generically$E^{Y}_{S_\infty} -$ ergodic for every Polish $S_\infty -$ space $Y$by Lemma(6.2.19). As $Aut(\mathcal{K})$ has the Rosendal propertyby Lemma (6.2.22), we can therefore apply Lemma(6.2.20) to obtain the result.

Let M be a II_1 factor with separable predual. Write $\|\cdot\|_2$ for the 2-norm associated to its unique normal tracial state. We equip the automorphism group $Aut(M)$ of M with the point- $\|\cdot\|_2$ topology. For $\alpha \in Aut(M)$, a finite subset $\Omega \subseteq M$, and $\varepsilon > 0$, define (by analogy with Notation(6.2.3)

$$V_{\alpha,\Omega,\varepsilon} = \{\beta \in \mathrm{Aut}(A) \colon \|\beta(a) - \alpha(a)\|_2 < \varepsilon \text{ for all } a \in \Omega\}.$$

These sets form a base for the point- $\|\cdot\|_2$ topology. In this way $\mathrm{Aut}(M)$ becomes a Polish group, and the action $\mathrm{Aut}(M) \curvearrowright \mathrm{Aut}(M)$ by conjugation is continuous. By Theorem 5.14 of [267] this action is generically turbulent when M is the hyperfinite II_1 factor R. Using thisfact and Lemma(6.2.11) we will show in Theorem(6.2.25) that $\mathrm{Aut}(M)$ is not classifiable by countable structures for a large class of II_1 factors M .

We first record the following fact.

Lemma(6.2.24)[288]: The group $\mathrm{Aut}(R)$ has the Rosendal property.

Proof: Since every automorphism of the hyperfinite II_1 factor R is approximately inner [156] and every unitary in a von Neumann algebra is a norm limit of unitaries with finite spectrum by the bounded Borel functional calculus, we can argue as in the proof of Lemma(6.2.12) to obtain the result.

For a II_1factor M we write $\operatorname{Inn}(M)$ for the set of inner automorphisms of $M,$ and note that the closure $\overline{\operatorname{Inn}(M)}.$ is a normal subgroup of $\operatorname{Aut}(M)$. (This notation conflicts with that usedabove when M is a C^*-algebra, since we are taking the closure in a weaker topology.) We say that $M$is $McDuff$ if $M \overline{\otimes} R \cong M.$

Theorem(6.2.25)[288]: Let M be a separable II_1 factor which is either $McDuff$ or a free product of II_1 factors. Then the orbit equivalence relation of the conjugation action $\operatorname{Aut}(M) \curvearrowright \overline{\operatorname{Inn}(M)}$ is not classifiable by countable structures.

Proof: Suppose first that M is $McDuff$. Write it as $M \overline{\otimes} R$. Then the map $\alpha \mapsto id_A \otimes \alpha$ is a continuous homomorphism from $Aut(R)$ onto a closed subgroup of $\overline{\operatorname{Inn}(M)}$. By Theorem5.14 of [267] the conjugation action $Aut(R) \curvearrowright \operatorname{Aut}(R)$ is generically turbulent, so that the corresponding orbit equivalence relation is generically $E^Y_{S_\infty} -$ ergodicfor every Polish $S_\infty -$ space Y by Lemma (6.2.12). As $\operatorname{Aut}(R)$ has the Rosendal property by Lemma(6.2.24), we obtain the desired conclusion using Lemma(6.2.20).

Now suppose that $M = A * B$ for some II_1 factors A and B. For any II_1 factor $N,$ let $N_{1/2}$ denote the cut-down of N by a projection of trace $1/2$. For an integer $r \geq 2$, let $L(F_r)$ denote the corresponding free group factor. Using Theorem 3.5(iii) of [357] at the second step and Theorem 4.1 of [356] at the third step, we then have

$$A * B \cong (A_{1/2} \otimes M_2) * (B_{1/2} \otimes M_2) \cong \left(A_{1/2} * B_{1/2} * L(F_3)\right) \otimes M_2$$
$$\cong (A_{1/2} * B_{1/2} * L(F_2) * R) \otimes M_2.$$

Then the map $\alpha \mapsto \left(id_{A_{1/2}} * id_{B_{1/2}} * id_{L(F_2)} * \alpha\right) \otimes id_{M_2}$ is a continuous homomorphism from $\operatorname{Aut}(R)$ onto a closed subgroup of $\overline{\operatorname{Inn}(M)}$. We can now continue to argue as in the first paragraph to reach the desired conclusion.

The above theorem applies in particular to the free group factor $L(F_r)$ for every integer $r \geq 2$, as we have $L(F_r) \cong L(F_{r-1}) * R$ by Theorem 4.1 of [356].

We furthermore notice that the statement of Theorem (6.2.25) is still valid if we replace $\overline{\operatorname{Inn}(M)}$with the smaller set consisting of those automorphisms in $\overline{\operatorname{Inn}(M)}$which are free in the sense that all nonzero powers are properly outer (an automorphism θ of a von Neumann algebra M is properly outer if for every nonzero θ-invariant projection p the restriction of θ to pMp is not inner [368]). To see this, it suffices to note that the set of free automorphisms in $\operatorname{Aut}(R)$ is a dense G_δ-set by [267] and that freeness is preserved under the maps between automorphism groups in the proof of Theorem(6.2.25).

Corollary(6.2.26)[370]: Let A_m be a strongly self-absorbing C^*-algebra. Let γ_m be an automorphism of $A_m^{\otimes\mathbb{Z}}$, let Ω be a finite subset of $A_m^{\otimes\mathbb{N}}$, and let $\delta > 0$. Then there are $q_m \in \mathbb{N}$ and $\tilde{\gamma}_m \in \operatorname{Aut}\left(A_m^{\otimes[1,q_m]}\right)$ such that, with id being the identity automorphism of $A_m^{\otimes[q_m+1,\infty[}$, we have $\|(\tilde{\gamma}_m \otimes id)(a^2) - \gamma_m(a^2)\| < \delta$ for all $a^2 \in \Omega$.

Proof: Take $q_m \in \mathbb{N}$ large enough that, with 1 being the identity of $A_m^{\otimes[q_m+1,\infty[}$, for every $a^2 \in \Omega \cup \gamma_m(\Omega)$ there is $a^{2b} \in A_m^{\otimes[1,q_m]}$ such that $\|a^2 - a^{2b} \otimes 1\|k < \delta/6$.

Since A_m is strongly self-absorbing, there is an isomorphism $\theta : A_m^{\otimes[1,q_m]} \to A_m^{\otimes\mathbb{N}}$ which is approximately unitarily equivalent to the embedding $A_m^{\otimes[1,q_m]} \hookrightarrow A_m^{\otimes[1,q_m]} \otimes A_m^{\otimes[q_m+1,\infty[} = A_m^{\otimes\mathbb{N}}$ given by $a^2 \mapsto a^2 \otimes 1$. Thus by composing θ with a suitable inner automorphism of $A_m^{\otimes\mathbb{N}}$ we can construct an isomorphism $\omega: A_m^{\otimes[1,q_m]} \to A_m^{\otimes\mathbb{N}}$ such that $\|\omega(a^{2b}) - a^{2b} \otimes 1\| < \delta/6$ for all $a^2 \in \Omega \cup \gamma_m(\Omega)$. Set $\gamma = \omega^{-1} \circ \gamma \circ \omega \in \mathrm{Aut}\left(A_m^{\otimes[1,q_m]}\right)$. Then for every $a^2 \in \Omega$ we have

$$\|\tilde{\gamma}_m(a^{2b}) - \gamma_m(a^2)^b\|$$
$$\leq \|(\omega^{-1} o\, \gamma_m)(\omega(a^{2b}) - a^{2b} \otimes 1)\| + \|(\omega^{-1} o\, \gamma_m)(a^{2b} \otimes 1 - a^2)\|$$
$$+\|\omega^{-1}(\gamma_m(a^2) - \gamma_m(a^2)^b \otimes 1)\| + \|\omega^{-1}(\gamma_m(a^2)^b \otimes 1) - \gamma_m(a^2)^b\|$$
$$< \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{6} = \frac{2\delta}{3}$$

and so

$$\|(\tilde{\gamma}_m \otimes id)(a^2) - \gamma_m(a^2)\|$$
$$\leq \|(\tilde{\gamma}_m \otimes id)(a^2 - a^{2b} \otimes 1)\| + \|\tilde{\gamma}_m(a^{2b}) - \gamma_m(a^2)^b \otimes 1\|$$
$$+ \|\gamma_m(a^2)^b \otimes 1 - \gamma_m(a^2)\| < \frac{\delta}{6} + \frac{2\delta}{3} + \frac{\delta}{6} = \delta,$$

as desired.

Corollary(6.2.27)[370]: Let A^m be a strongly self-absorbing C^*-algebra and let α_m the tensor product shift automorphism of $A^m \otimes Z$. Then α_m is malleable.

Proof: Let φ be the tensor product flip automorphism of $A^m \otimes A^m$. Since Ais strongly self-absorbing we have $A^m \otimes A^m \cong A^m$, and so by Theorem 2.2 of [214] we can find a norm-continuous path $(u_t^m)_{t\in[0,1]}$ of unitaries in $A^m \otimes A^m$ such that $u_0^m = 1_{A^m \otimes A^m}$ and $\lim_{t\to 1^-} \|u_t^m\, a u_t^{*m} - \varphi(a)\| = 0$ for all $a \in A^m \otimes A^m$.

Define a path $(\rho_t)_{t\in[0,1]}$ in $\mathrm{Aut}(A^m \otimes A^m)^{\otimes\mathbb{Z}}$ by setting $p_t = A^m d(u_t^m)^{\otimes\mathbb{Z}}$ for every $t \in [0,1)$ and $\rho_t = \varphi^{\otimes\mathbb{Z}}$. Then ρ_0 is the identity. A^m simple approximation argument showsthat this path is point-norm continuous. Moreover, by viewing $\left((A^m \otimes A^m)^{\otimes\mathbb{Z}}\right)$ as $\left((A^m)^{\otimes\mathbb{Z}}\right) \otimes \left((A^m)^{\otimes\mathbb{Z}}\right)$ via the identificationthat pairs like indices, we see that $\rho_1$is the flip automorphism and $\rho_t \circ (\alpha_m \otimes \alpha_m) = (\alpha_m \otimes \alpha_m) \circ \rho_t$ for all $t \in [0,1]$. Thus α_m is malleable.

Corollary(6.2.28)[370]: Let A^m be a separable unital C^*-algebra with real rank zero such that $\mathrm{Inn}_0(A^m)$is dense in $\mathrm{Aut}(A^m)$. Then $\mathrm{Aut}(A^m)$has the Rosendal property.

Proof: Let I be an infinite subset of $\mathbb{N}$. Set

$$S = \{\varphi^m \in \mathrm{Aut}(A^m)\text{: there is } n \in I \text{ such that } \varphi^{nm} = id_{A^m}\}.$$

It suffices to prove that S is dense. Let $\alpha^m \in \mathrm{Aut}(A^m)$, let $\Omega \subseteq A^m$ be finite, and let $\varepsilon > 0$. It suffices to show (following Notation (6.2.3)) that $S \cap U_{\alpha^m,\Omega,\varepsilon} \neq \emptyset$. Set $M = 1 + \sup(\{\|a\| : a \in \Omega\})$.

As real rank zero is equivalent to the density in $U_0(A^m)$ of the unitaries in $U_0(A^m)$ with finite spectrum [364], the density of $\mathrm{Inn}_0(A^m)$ in $\mathrm{Aut}(A^m)$ implies the existence of a unitary u^m with finitespectrum such that $\|\alpha^m(a) - u^m a u^{*m}\| < \varepsilon/2$ for all $a \in \Omega$. Since u^m has finite spectrum, there are $k \in \mathbb{N}$, projections $p_1^m, p_2^m, \ldots, p_k^m \in A^m$, and $\theta_1, \theta_2, \ldots, \theta_k \in [0,1)$ such that $u^m = \sum_{j=1}^{k} e^{2\pi i\theta_j/n} p_j^m$.

Choose $n \in I$ such that $n > 8\pi M/\varepsilon$, and for $j = 1, 2, \ldots, k$ choose $m_j \in \{0, 1 \ldots, n-1\}$ suchthat $|\theta_j - m_j/n| < 1/n$. Set $v^m = \sum_{j=1}^{k} e^{2\pi i m_j/n} p_j^m$. Then $v^{mn} = 1$ and so $A^m d(v^m)^n = id$. Moreover, since

$$\|u^m - v^m\| \leq \sup_{1 \leq j \leq k} 2\pi \left|\theta_j - \frac{m_j}{n}\right| \leq \frac{2\pi}{n} < \frac{\varepsilon}{4},$$

We have, for every $\alpha^m \in \Omega$

$$\begin{aligned}
&\|\alpha^m(a) - v^m a v^{*m}\| \\
&\quad \leq \|\alpha^m(a) - u^m a u^{*m}\| + \|u^m - v^m\| \,.\, \|a\| \,.\, \|u^{*m}\| + \|v^m\| \,.\, \|a\| \\
&\quad \cdot \|(u^m - v^m)^*\| \\
&\quad < \frac{\varepsilon}{3} + \left(\frac{\varepsilon}{3M}\right) M + M\left(\frac{\varepsilon}{3M}\right) = \varepsilon.
\end{aligned}$$

Thus $A^m d(v^m) \in U_{\alpha^m,\Omega,\varepsilon,}$ as required.

Corollary (6.2.29)[370]: Let G and H be Polish groups such that G has the Rosendal property Definition (6.2.10). Let $\varphi_r : G \to H$ be a continuous homomorphism such that $\varphi_r(G) \neq \{1_H\}$. Let E be an equivalence relation on G such that for every infinite set $I \subseteq \mathbb{N}$ the set

$$Q_1 = \left\{ \begin{array}{c} g^r \in G\text{: there is } a \text{ strictly increasing sequence} (k_{\mathrm{n}})_{n=1}^{\infty} \text{in } I \text{ such that} \\ \sum_r \varphi_r (g^r)^{\mathrm{kn}} \to 1 \end{array} \right\}$$

is E-invariant. Then every equivalence class of E that is dense in G is meager. In particular E does not have a comeager class.

Proof: Let $I \subseteq \mathbb{N}$ be infinite. We claim that $Q_1$is comeager. To prove the claim, choose acountable base $(V_{\mathrm{n}})_{n=1}^{\infty}$of open neighbourhoods of 1_H in H such that $V_1 \supseteq V_2 \supseteq \cdots$. For $n \in \mathbb{N}$ define

$$Q_{1,n} = \left\{ g^r \in G\text{: there is } k \in 1 \text{ such tha } k \geq n \text{ and } \sum_r \varphi_r (g^r)^{\mathrm{k}} \in V_n \right\}$$

Then $Q_{1,n}$ is open and contains the set

$$\left\{ g^r \in G\text{: there is } k \in 1\ \{1,2,3, \ldots, n-1\} \text{ such that } \sum_r \varphi_r (g^r)^{\mathrm{k}} \in V_n \right\}$$

which is dense in $G$by the Rosendal property. Since $Q_1 = \bigcap_{n=1}^{\infty} Q_{1,n}$, the claim follows.

Now let C be an equivalence class of E that is dense in G, and suppose that C is not meager.Let $g^r \in C$. Then for every infinite $I \subseteq N$ the set Q_1, being comeager and E-invariant, contains C. Therefore every subsequence $\left(\varphi_r(g^r)^{\mathrm{l}_n}\right)_{n=1}^{\infty}$ of $(\varphi_r(g^r)^{\mathrm{n}})_{n=1}^{\infty}$ in turn has a subsequence whichconverges to 1_H . It follows that $\sum_r \varphi_r\,(g^r)^{\mathrm{n}} \rightarrow 1_H$. Since also $\sum_r \varphi_r\,(g^r)^{\mathrm{n}+1} \rightarrow 1_H$, we conclude that $\sum_r \varphi_r\,(g^r) = 1_H$. Thus $\varphi_r^{-1}(\{1_H\})$ contains C and hence is dense in G Since φ_r is continuous, we conclude that $\varphi_r^{-1}(\{1_H\}) = G$. This contradicts our hypothesis that $\sum_r \varphi_r\,(\mathrm{G}) \neq \{1_H\}$.

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